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A locking-free finite difference method on staggered grids for linear elasticity problems☆

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ABSTRACT

A finite difference method on staggered grids is constructed on general nonuniform rectangular partition for linear elasticity problems. Stability, optimal-order error estimates in discrete H^1-norms on general nonuniform grids and second-order superconvergence on almost uniform grids have been obtained. These theoretical results are uniform about the *Lamé* constant $\lambda \in (0, \infty)$ so the finite difference method is locking-free. The method and theoretical results can be extended to three dimensional problems. Numerical experiments using the method show agreement of the numerical results with theoretical analysis.

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1. Introduction

The analysis of the deformation of compressible and nearly incompressible linear elastic solids has applications in many important physical problems. Numerical methods for the compressible and nearly incompressible linear elasticity equation plays an important role in solid mechanics. Different kinds of numerical methods have been introduced to numerically solve the linear elasticity problems. It is well known that the low-order finite element methods are the common methods applied to compressible linear elasticity problems [1,2]. However, when the material is nearly incompressible, that is, the Poisson ratio $\nu \to \frac{1}{2}$ (or the Lamé constant $\lambda \to \infty$), the accuracy deteriorates. This phenomenon is known as the "locking", which means that the approximate solutions do not converge uniformly about λ. For the conforming finite element methods, Babuška and Suri [3] found that locking cannot be avoided on quadrilateral meshes for any polynomial of degree $k \geq 1$.

To overcome these problems, many numerical methods have been proposed, such as nonconforming finite element methods [4,5], h—version finite elements [3,6], the B-bar method [1], mixed element methods [7–10], enhanced assumed strain methods [11–13], the stream function approach [14], the reduced integration stabilization method [15], and the mimetic finite difference method [16]. There are also several publications investigating an average nodal pressure formulation in which a constant pressure field is enforced [17–20].

Among these methods, only a few are based on rigorous mathematical analysis, for instance, [10,20–22]. Recent years, some new methods with a complete mathematical analysis were proposed for nearly incompressible elasticity problems, see for example, [23–26].

The finite difference method on staggered grids has been one of the hot research topics in scientific computing and numerical analysis, which has been used to approximate the solutions of fluid problems, elliptic and porous media flow problems and Maxwell equations. The finite difference method on staggered rectangular or cuboid grids for fluid problems,

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called marker and cell (MAC), has been used for a long time, see, for example, [27–33]. It has been widely used in engineering applications and has been shown to locally conserve the mass, momentum and kinetic energy [34,35]. The theoretical analysis has been given in [36–40] which proved that the MAC scheme has first order convergence for both the velocity and the pressure on uniform rectangular meshes. The second-order superconvergence analysis has been considered in [41] and [42] recently. There are many publications to consider the finite difference methods on staggered grids, called block-centered finite difference, for elliptic, parabolic and porous media flow problems. See, for example, [43–46]. Finite difference method on staggered grids to solve the Maxwell problems, the famous Yee's scheme [47], has been introduced for a long time, and been studied deeply, for example, in [48–52].

In this paper, we first construct the finite different scheme on staggered grids for the linear elasticity problem on non-uniform rectangular grids. The scheme is obtained based on rectangular partition with the x-component of displacement approximated at the midpoint of vertical edges of the rectangular cell, and the y-component of displacement approximated at the midpoint of horizontal edges of the cell. Then we establish the rigorous mathematical analysis. We give the stability theorem which holds for all $\lambda \in (0, \infty)$. Then inspired by the analysis technique in [43,45,49] for elliptic, Darcy–Forchheimer or Maxwell's equations, we give the first order error estimates for the displacement in discrete H^1 norms on non-uniform grids. On almost uniform grids second-order superconvergence is established for the displacement in discrete H^1 norms. All the stability, the error estimate and superconvergence results hold uniformly about $\lambda \in (0, \infty)$ then the presented scheme is locking-free. The finite difference method and the theoretical results can been extended to three problems without any difficulty. Finally some numerical experiments are carried out, which show that the numerical results are consistent with the theoretical analysis.

The paper is organized as follows. In Section 2 we give the problem, the finite difference scheme on staggered grids and its stability. In Section 3 we present the numerical analysis for the scheme. In Section 4 some numerical experiments using the scheme are carried out.

Through out the paper we use C, with or without subscript, to denote a positive constant, which could have different values at different appearances. We use $\|\cdot\|_{r,e}$, $|\cdot|_{r,e}$ to denote the norm, semi-norm of the Sobolev space $H^r(e)$ or $(H^r(e))^2$, respectively, use $\|\cdot\|_{r,\infty,e}$ to denote the norm of $W^{r,\infty}(e)$ or $(W^{r,\infty}(e))^2$ and omit the subscript e if $e = \Omega$.

2. The finite difference method on staggered grids

In this section we consider the finite difference on staggered grids for the elasticity problems with homogeneous boundary condition. For simplicity we deal with the problems in a two dimensional domain Ω. The scheme and the theoretical analysis can been extended to three dimensional problems easily.

Denote the displacement by $\boldsymbol{u} = (u^x, u^y)^T$, the elasticity problem in Ω with homogeneous boundary condition is described by the following model.

$$\begin{cases} -(\lambda+\mu)\nabla(\nabla\cdot\boldsymbol{u}) - \mu\Delta\boldsymbol{u} = \boldsymbol{f} & \text{in } \Omega, \\ \boldsymbol{u} = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

Here, $\lambda > 0$ and $\mu > 0$ are *Lamé* constants, $\nabla\cdot\boldsymbol{u} = \frac{\partial u^x}{\partial x} + \frac{\partial u^y}{\partial y}$, $\boldsymbol{f} = (f^x, f^y)^T \in (L^2(\Omega))^2$ is a term related to body forces.

When the material is nearly incompressible, that is, the Poisson ratio $\nu \to \frac{1}{2}$ (or the Lamé constant $\lambda \to \infty$), the approximate solutions do not converge uniformly about λ. This phenomenon is called "Poisson locking" in the literature. To overcome the locking effects, the most popular method is the mixed method by introducing the pressure $p = \lambda\nabla\cdot\boldsymbol{u}$ as an independent unknown. Then the problem can be described by the following equations:

$$\begin{cases} -\dfrac{\lambda+\mu}{\lambda}\dfrac{\partial p}{\partial x} - \mu\left(\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}\right)u^x = f^x & \text{in } \Omega, \\ -\dfrac{\lambda+\mu}{\lambda}\dfrac{\partial p}{\partial y} - \mu\left(\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}\right)u^y = f^y & \text{in } \Omega, \\ \dfrac{\partial u^x}{\partial x} + \dfrac{\partial u^y}{\partial y} - \dfrac{p}{\lambda} = 0 & \text{in } \Omega, \\ \boldsymbol{u} = (u^x, u^y)^T = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.2}$$

We construct the finite difference scheme on staggered grids for the above model problem. For simplicity we suppose $\Omega = (0, a) \times (0, b)$ and use the partitions and notations as follows.

The two dimensional domain Ω is partitioned by $\delta_x \times \delta_y$, where

$$\delta_x : 0 = x_0 < x_1 < \cdots < x_{n_x} = a, \quad \delta_y : 0 = y_0 < y_1 < \cdots < y_{n_y} = b.$$

For simplicity we also use the following notations:

$$\begin{cases} x_{-1/2} = x_0 = 0, \ \ x_{n_x+1/2} = x_{n_x} = a, \\ y_{-1/2} = y_0 = 0, \ \ y_{n_y+1/2} = y_{n_x} = b. \end{cases}$$

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