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A new regularity criterion of the 2D MHD equations

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ABSTRACT

In this paper, we establish a new regularity criterion for the two-dimensional incompressible generalized magnetohydrodynamics equations.

1. Introduction and main results

This note is concerned with the following two-dimensional (2D) generalized magnetohydrodynamics (GMHD) equations:

$$\begin{cases} \partial_t u + (u\cdot\nabla)u + (-\Delta)^{\alpha}u + \nabla p = (b\cdot\nabla)b, \quad x\in\mathbb{R}^2,\ t>0,\\ \partial_t b + (u\cdot\nabla)b + (-\Delta)^{\beta}b = (b\cdot\nabla)u,\\ \nabla\cdot u = 0, \quad \nabla\cdot b = 0,\\ u(x,0)=u_0(x), \qquad b(x,0)=b_0(x), \end{cases} \tag{1.1}$$

where $\alpha\in[0,\ 2]$ and $\beta\in[0,\ 2]$ are real parameters, $u=u(x,t)=(u_1(x,\ t),\ u_2(x,\ t))$, $b=b(x,t)=(b_1(x,\ t),\ b_2(x,\ t))$ and $p=p(x,t)$ denote the velocity vector, the magnetic vector and pressure scalar fields, respectively. For simplicity, we denote $\Lambda^{\gamma} := (-\Delta)^{\frac{\gamma}{2}}$, which is defined via the Fourier transform as $\widehat{\Lambda^{\gamma}f}(\xi) = |\xi|^{\gamma}\hat{f}(\xi)$. The GMHD equations play a fundamental role in geophysics, astrophysics, cosmology and engineering (see, e.g., [1]). We point out the convention that by $\alpha=0$ we mean that there is no dissipation in $(1.1)_1$, and similarly $\beta=0$ represents that there is no diffusion in $(1.1)_2$.

Many important contributions have been made on the well-posedness result for the 2D GMHD equations (1.1), and we list only some results relevant to our concerns (see [2–9] with no intention to be complete). It is worthwhile to point out that the latest global regularity results of the 2D GMHD equations (1.1) can be summarized as

$$(1)\ \alpha>0,\ \beta=1; \qquad (2)\ \alpha=0,\ \beta>1; \qquad (3)\ \alpha=2,\ \beta=0,$$

see [2,4,5] for details (one also refers to [10,8,9] for logarithmic type dissipation). To the best of our knowledge, apart from the above mentioned cases, the global regular result for the remainder cases is not known up to date. Therefore, it is interesting to consider regularity criteria (see [11–15,7,16–18]). The target of this paper is to establish a new scaling invariant regularity criterion for the system (1.1) with the special case $\alpha=\beta$. More precisely, our main results read as follows:

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Theorem 1.1. *Consider the system (1.1) with $0<\alpha=\beta<1$ and assume $(u_0,b_0)\in H^s(\mathbb{R}^2)\times H^s(\mathbb{R}^2)$ with $s>2$. Let (u,b) be a local smooth solution to the system (1.1) with the initial data. If the following condition holds*

$$\int_0^T \frac{\|T(\nabla u,\nabla b)(t)\|_{\dot{B}^0_{\infty,\infty}}^{\frac{1}{2}}}{\left(\ln e+\|u(t)\|_{H^s}^2+\|b(t)\|_{H^s}^2\right)^{\frac{3}{4}}}\,dt<\infty, \tag{1.2}$$

then (u,b) can be extended beyond time T, where $T(\nabla u,\nabla b)$ is given by

$$T(\nabla u,\nabla b)=2\partial_1 b_1(\partial_2 u_1+\partial_1 u_2)-2\partial_1 u_1(\partial_2 b_1+\partial_1 b_2). \tag{1.3}$$

Remark 1.2. At present, we are not able to prove that Theorem 1.1 is true for more general $\alpha\neq\beta$. The essential reason is that we need the nice structure of the system (1.1) with $\alpha=\beta$, namely, the two quantities G_1 and G_2 in (2.3) introduced in [19]. Finally, whether Theorem 1.1 holds true for the completely inviscid case ($\alpha=\beta=0$) also remains unknown. However, it follows from (2.4) and (2.7) that for the system (1.1) with $\alpha=\beta=0$, the following condition

$$\int_0^T \|T(\nabla u,\nabla b)(t)\|_{L^\infty}\,dt<\infty$$

implies that (u,b) can be extended beyond time T. According to the scaling invariant argument, the following is actually a scaling invariant

$$\int_0^T \|T(\nabla u,\nabla b)(t)\|_{L^\infty}^{\frac{1}{2}}\,dt<\infty.$$

Unfortunately, it is not clear to prove it.

2. The proof of Theorem 1.1

Throughout the paper, C stands for some real positive constants which may be different in each occurrence and $C(a)$ denotes the positive constant depending on a.

The basic L^2-energy estimate shows that

$$\|u(t)\|_{L^2}^2+\|b(t)\|_{L^2}^2+2\int_0^t(\|\Lambda^\alpha u(\tau)\|_{L^2}^2+\|\Lambda^\alpha b(\tau)\|_{L^2}^2)\,d\tau\le\|u_0\|_{L^2}^2+\|b_0\|_{L^2}^2.$$

Taking curls on the GMHD equations (1.1), it follows that the vorticity $\omega:=\partial_{x_1}u_2-\partial_{x_2}u_1$ and the current density $j:=\partial_{x_1}b_2-\partial_{x_2}b_1$ satisfy

$$\partial_t\omega+(u\cdot\nabla)\omega+\Lambda^{2\alpha}\omega-(b\cdot\nabla)j=0, \tag{2.1}$$

$$\partial_t j+(u\cdot\nabla)j+\Lambda^{2\alpha}j-(b\cdot\nabla)\omega=T(\nabla u,\nabla b), \tag{2.2}$$

where $T(\nabla u,\nabla b)$ is given by (1.3). Following [19], we introduce

$$G_1:=\omega+j,\qquad G_2:=\omega-j, \tag{2.3}$$

which obey

$$\partial_t G_1+(u\cdot\nabla)G_1+\Lambda^{2\alpha}G_1-(b\cdot\nabla)G_1=T(\nabla u,\nabla b), \tag{2.4}$$

$$\partial_t G_2+(u\cdot\nabla)G_2+\Lambda^{2\alpha}G_2+(b\cdot\nabla)G_2=-T(\nabla u,\nabla b). \tag{2.5}$$

Multiplying Eqs. (2.4) and (2.5) by $|G_1|^{p-2}G_1$ and $|G_2|^{p-2}G_2$, respectively, and using the fact

$$\int_{\mathbb{R}^2}\Lambda^{2\alpha}G_1(|G_1|^{p-2}G_1)\,dx\ge 0,\qquad \int_{\mathbb{R}^2}\Lambda^{2\alpha}G_2(|G_2|^{p-2}G_2)\,dx\ge 0,$$

we have

$$\begin{aligned}
&\frac{1}{p}\frac{d}{dt}(\|G_1(t)\|_{L^p}^p+\|G_2(t)\|_{L^p}^p)\\
&\le\int_{\mathbb{R}^2}|T(\nabla u,\nabla b)|\,|G_1|^{p-1}\,dx+\int_{\mathbb{R}^2}|T(\nabla u,\nabla b)|\,|G_2|^{p-1}\,dx\\
&\le C\|T(\nabla u,\nabla b)\|_{L^p}(\|G_1\|_{L^p}^{p-1}+\|G_2\|_{L^p}^{p-1})\\
&=C\left\||T(\nabla u,\nabla b)|^{\frac{1}{2}}|T(\nabla u,\nabla b)|^{\frac{1}{2}}\right\|_{L^p}(\|G_1\|_{L^p}^{p-1}+\|G_2\|_{L^p}^{p-1})
\end{aligned}$$

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