ARTICLE IN PRESS

Computers and Mathematics with Applications ▮ (▮▮▮▮) ▮▮▮–▮▮▮

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

Pseudo-spectral least squares method for linear elasticity

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ARTICLE INFO

Article history:
Received 29 January 2018
Received in revised form 6 May 2018
Accepted 17 June 2018
Available online xxxx

Keywords:
Linear elasticity
Least squares method
Legendre and Chebyshev pseudo-spectral method

ABSTRACT

The first order system least squares Legendre and Chebyshev spectral method for two dimensional space linear elasticity is investigated. The drilling rotation is defined as a new variable and the linear elasticity equation is supplemented with an auxiliary equation. The weighted L^2-norm least squares principle is applied to a stress–displacement–rotation. It is shown that the homogeneous least squares functional is equivalent to weighted H^1-norm like for stress and weighted H^1-norm for displacement and rotation. This weighted H^1-norm equivalence is λ-uniform. Spectral convergence for both Legendre and Chebyshev approaches are given along with some numerical experiments. The generalization for three dimensional spaces is also provided.

1. Introduction

In this paper, we consider the linear elasticity problem in two dimension space which consists of the constitutive and equilibrium equations of the form

$$\begin{cases} \mathcal{E}(\mathbf{u}) - \boldsymbol{\ell}\,\boldsymbol{\sigma} = 0, & \text{in } \Omega, \\ \nabla \cdot \boldsymbol{\sigma} + \mathbf{f} = 0, & \text{in } \Omega, \\ \mathbf{u} = \mathbf{0}, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\boldsymbol{\sigma}$ is stress tensor, $\mathcal{E}(\mathbf{u})$ is strain tensor, $\mathbf{f}$ is body force and $\mathbf{u}$ is displacement. Here $\Omega \subseteq \mathbf{R}^2$ is an open bounded domain with boundary $\partial\Omega$. Denote the Lamé constants by

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}, \quad \text{and} \quad \mu = \frac{E}{2(1+\nu)},$$

where $E > 0$ is the modulus of elasticity, $\nu \in (-1, 1/2)$ is the Poisson ratio of the elastic material. We also have

$$\boldsymbol{\ell} = \frac{1}{2\mu}\Big(\mathbf{I} - \frac{\lambda}{2\lambda+2\mu}\mathbf{b}\mathbf{b}^t\Big), \text{ with } \quad \mathbf{b} = (1,\ 0,\ 0,\ 1)^t,$$

and

$$\boldsymbol{\ell}\boldsymbol{\sigma} = \frac{1}{2\mu}\begin{bmatrix} \frac{2\mu+\lambda}{2\lambda+2\mu}\boldsymbol{\sigma}_{11} - \frac{\lambda}{2\lambda+2\mu}\boldsymbol{\sigma}_{22} & \boldsymbol{\sigma}_{12} \\ \boldsymbol{\sigma}_{12} & -\frac{\lambda}{2\lambda+2\mu}\boldsymbol{\sigma}_{11} + \frac{2\mu+\lambda}{2\lambda+2\mu}\boldsymbol{\sigma}_{22} \end{bmatrix}.$$

Numerical approximation of the linear elasticity is a challenging problem in the numerical analysis community especially when the material tends to become incompressible, in the other words, the Lamé constant λ tends to infinity for fixed Lamé

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https://doi.org/10.1016/j.camwa.2018.06.028
0898-1221/

constant μ, or the Poisson ratio ν tends to 0.5. A plethora of methods have been investigated to approximate the solution of (1.1). The linear finite element method [1], the nonconforming mixed multigrid method [2], the discontinuous Galerkin methods [3], and virtual elements [4] to name a few.

Least squares finite element methods [5–12] have also been applied to approximate the solution of linear elasticity (1.1). The authors in [6,7] introduced the displacement flux as a new variable and two-stage algorithms used to approximate the solution of the least squares functional defined based on L^2-norm. They first solve the displacement gradient and then solve the displacement itself. The least squares method based on a discrete minus one inner product is also investigated in [5]. Cai and Starke used the first order system (1.1) supplemented with the symmetry of the stress tensor $\boldsymbol{\sigma} = \boldsymbol{\sigma}^t$ in [9]. This approach does not apply for incompressible materials and requires effective discretizations and efficient solvers for the pure displacement problem when materials are nearly incompressible. They removed symmetry constraint of stress in [10] and showed the ellipticity and continuity of the least squares method. The authors in [12,13] modified the least squares functional in [10] to improve the momentum balance. In the above works, the variables of interest are displacement and stress. Jiang and Wu [11] introduced drilling rotation as a new variable and provided some numerical experiments.

The aim of this paper is to investigate the first order system least squares method for the pure displacement problem in linear elasticity based on displacement–stress–rotation formulation. However, instead of applying the finite element approach, pseudo-spectral Legendre and Chebyshev methods are used. Pseudo-spectral least squares methods have received much attention recently and have been applied to different partial differential equations [14–19]. In this work we combine the spectral accuracy of the pseudo-spectral method [20] and the least squares approach [21] with its advantages to approximate the solution of (1.1). To this end, following the idea of [11], we define the drilling rotation which is the same as vorticity in fluid mechanic, as a new variable. The first order system Eq. (1.1) is then supplemented with the drilling rotation definition and another first order equation containing the derivative of the rotation. The least squares functional is defined to be the sum of the L^2_ω-norm of residual of the first order system. The least squares functional, under H^2_ω regularity assumption, is equivalent to H^1_ω-norm like for stress and H^1_ω-norm for displacement and rotation. It should be noted that this H^1_ω-norm equivalence is λ-uniform, which guarantees the spectral accuracy of the method independent of λ. Spectral convergence of the method for both Legendre and Chebyshev along with some numerical examples are shown.

The content of this paper is organized as follows. In Section 2, we provide preliminaries that will be used in subsequent sections. In Section 3, least squares functional is defined and its well-posedness is proven. Section 4 is devoted to discrete least squares Legendre and Chebyshev methods as well as their spectral convergence. The method is illustrated numerically in Section 5. The extension for three dimensional spaces is provided in Section 6. We conclude the paper in Section 7.

2. Preliminaries

Preliminaries are provided in this section, briefly. We refer the reader to [14,15,22] for more details. We use the standard notations and definitions for the weighted Sobolev spaces $H^s_w(\mathbf{D}), s \geq 0$ where $\mathbf{D} = [-1, 1]^2$. The weight function is $w(\mathbf{x}) = \hat{w}(x)\hat{w}(y)$ where $\hat{w}(t) = 1$ is the Legendre weight function and $\hat{w}(t) = \frac{1}{\sqrt{1-t^2}}$ is the Chebyshev weight function. The space $H^0_w(\mathbf{D})$ indicates $L^2_w(\mathbf{D})$, in which the norm and inner product will be denoted by $\|\cdot\|_{w,\mathbf{D}}$ and $(\cdot, \cdot)_{w,\mathbf{D}}$, respectively. Let $H^1_{0,w}(\mathbf{D})$ be the subspace of $H^1_w(\mathbf{D})$, consisting of the functions which vanish on the boundary. For the Legendre case, we will simply write the notations without the subscripts w. Denote by $H^{-1}_w(\mathbf{D})$ the dual space of the space $H^1_{0,w}(\mathbf{D})$ equipped with its norm

$$\|u\|_{-1,w,\mathbf{D}} := \sup_{\phi \in H^1_{0,w}(\mathbf{D})} \frac{(u,\phi)_{w,\mathbf{D}}}{\|\phi\|_{1,w,\mathbf{D}}}. \tag{2.1}$$

Let

$$H_w(\text{div}, \mathbf{D}) = \{\mathbf{v} \in L^2_w(\mathbf{D})^2 \;:\; \nabla \cdot \mathbf{v} \in L^2_w(\mathbf{D})\},$$

$$H_\omega(\text{curl}, \mathbf{D}) = \{\mathbf{v} \in L^2_\omega(\mathbf{D})^2 \;:\; \nabla \times \mathbf{v} \in L^2_\omega(\mathbf{D})\},$$

which are Hilbert spaces under the respective norm

$$\|\mathbf{v}\|_{\omega,\text{div},\mathbf{D}} = \left(\|\mathbf{v}\|^2_{\omega,\mathbf{D}} + \|\nabla \cdot \mathbf{v}\|^2_{\omega,\mathbf{D}}\right)^{1/2},$$

and

$$\|\mathbf{v}\|_{\omega,\text{curl},\mathbf{D}} = \left(\|\mathbf{v}\|^2_{\omega,\mathbf{D}} + \|\nabla \times \mathbf{v}\|^2_{\omega,\mathbf{D}}\right)^{1/2}.$$

Denote the space of all polynomials of degree less than or equal to N by $\mathcal{P}_N$. Let $\{\xi_i\}_{i=0}^N$ be the Legendre–Gauss–Lobatto (LGL) or Chebyshev–Gauss–Lobatto (CGL) points on $[-1, 1]$ such that $-1 =: \xi_0 < \xi_1 < \cdots < \xi_{N-1} < \xi_N := 1$, with the corresponding quadrature weights $\{w_i\}_{i=0}^N$. For Legendre case, $\{\xi_i\}_{i=0}^N$ are the zeros of $(1-t^2)L'_N(t)$ where L_N is the Nth Legendre polynomial and the corresponding quadrature weights $\{w_i\}_{i=0}^N$ are given by

$$w_0 = w_N = \frac{2}{N(N+1)}, \quad w_j = \frac{2}{N(N+1)}\frac{1}{[L_N(\xi_j)]^2}, \quad 1 \leq j \leq N-1.$$