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Several new regularity criteria for the axisymmetric Navier–Stokes equations with swirl

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ABSTRACT

In this paper, we consider the axisymmetric Navier–Stokes equations with swirl, and show that the global regularity is ensured if we add some (weighted) integrable conditions on $\omega^\theta = \partial_z u^r - \partial_r u^z$, $\partial_r u^r$, $\partial_z u^z$ or $\partial_r u^\theta$.

1. Introduction

This paper studies the three-dimensional Navier–Stokes equations, which in Eulerian coordinates read

$$\begin{cases} \partial_t \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - \triangle \boldsymbol{u} + \nabla \Pi = \boldsymbol{0}, \\ \nabla\cdot\boldsymbol{u} = \boldsymbol{0}, \\ \boldsymbol{u}(0) = \boldsymbol{u}_0, \end{cases} \tag{1}$$

where

$$\boldsymbol{u} = (u^1, u^2, u^3) = u^1\boldsymbol{e}_1 + u^2\boldsymbol{e}_2 + u^3\boldsymbol{e}_3$$

is the fluid velocity field, Π is a scalar pressure, and $\boldsymbol{u}_0$ is the prescribed initial data satisfying the compatibility condition $\nabla\cdot\boldsymbol{u}_0 = 0$ in the distributional sense.

In the last century, Leray [1] and Hopf [2] have proven that (1) possesses at least one global weak solution for initial data of finite energy. However, the issue of its regularity and uniqueness is a challenging open problem in mathematical fluid dynamics. Pioneered by Serrin [3] and Prodi [4], there exist many sufficient conditions to ensure the smoothness of the solution. Particularly, we have the following regularity criterion (see [3–5] for example)

$$\boldsymbol{u} \in L^\alpha(0,T;L^\beta(\mathbb{R}^3)), \quad \frac{2}{\alpha} + \frac{3}{\beta} = 1, \quad 3 \leqslant \beta \leqslant \infty. \tag{2}$$

In this paper, we shall study the axisymmetric solutions of (1). A solution is axisymmetric if the velocity field can be represented as

$$\boldsymbol{u} = u^r(t,r,z)\boldsymbol{e}_r + u^z(t,r,z)\boldsymbol{e}_\theta + u^z(t,r,z)\boldsymbol{e}_z, \tag{3}$$

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where

$$\begin{aligned} \boldsymbol{e}_r &= \left(\frac{x_1}{r}, \frac{x_2}{r}, 0\right) = (\cos\theta, \sin\theta, 0)\,, \\ \boldsymbol{e}_\theta &= \left(-\frac{x_2}{r}, \frac{x_1}{r}, 0\right) = (-\sin\theta, \cos\theta, 0)\,, \\ \boldsymbol{e}_z &= (0, 0, 1) \end{aligned}$$

are the basis of $\mathbb{R}^3$ in the cylindrical coordinates. Here, u^r, u^θ and u^z are called the radial, swirl (or azimuthal) and axial components of $\boldsymbol{u}$ respectively. Thus the system (1) can be equivalently reformulated as

$$\begin{cases} \frac{\tilde{D}}{Dt}u^r - \left(\partial_r^2 + \partial_z^2 + \frac{1}{r}\partial_r - \frac{1}{r^2}\right)u^r - \frac{(u^\theta)^2}{r} + \partial_r \Pi = 0, \\ \frac{\tilde{D}}{Dt}u^\theta - \left(\partial_r^2 + \partial_z^2 + \frac{1}{r}\partial_r - \frac{1}{r^2}\right)u^\theta + \frac{u^r u^\theta}{r} = 0, \\ \frac{\tilde{D}}{Dt}u^z - \left(\partial_r^2 + \partial_z^2 + \frac{1}{r}\partial_r\right)u^z + \partial_z \Pi = 0, \\ \partial_r(ru^r) + \partial_z(ru^z) = 0, \\ (u^r, u^\theta, u^z)(0) = (u_0^r, u_0^\theta, u_0^z), \end{cases} \tag{4}$$

where

$$\frac{\tilde{D}}{Dt} = \partial_t + u^r\partial_r + u_z\partial_z. \tag{5}$$

Taking curl of $(1)_1$ and denoting by

$$\boldsymbol{\omega} = \nabla \times \boldsymbol{u} = \omega^r \boldsymbol{e}_r + \omega^\theta \boldsymbol{e}_\theta + \omega^z \boldsymbol{e}_z \tag{6}$$

with

$$\omega^r = -\partial_z u^\theta, \quad \omega^\theta = \partial_z u^r - \partial_r u^z, \quad \omega^z = \partial_r u^\theta + \frac{u^\theta}{r}, \tag{7}$$

we may rewrite the vorticity equation

$$\partial_t \boldsymbol{\omega} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{\omega} - (\boldsymbol{\omega}\cdot\nabla)\boldsymbol{u} - \triangle\boldsymbol{\omega} = \boldsymbol{0}, \tag{8}$$

as

$$\begin{cases} \frac{\tilde{D}}{Dt}\omega^r - \left(\partial_r^2 + \partial_z^2 + \frac{1}{r}\partial_r - \frac{1}{r^2}\right)\omega^r - (\omega^r\partial_r + \omega^z\partial_z)u^r = 0, \\ \frac{\tilde{D}}{Dt}\omega^\theta - \left(\partial_r^2 + \partial_z^2 + \frac{1}{r}\partial_r - \frac{1}{r^2}\right)\omega^\theta - \frac{2u^\theta\partial_z u^\theta}{r} - \frac{u^r\omega^\theta}{r} = 0, \\ \frac{\tilde{D}}{Dt}\omega^z - \left(\partial_r^2 + \partial_z^2 + \frac{1}{r}\partial_r\right)\omega^z - (\omega^r\partial_r + \omega^z\partial_z)u^z = 0. \end{cases} \tag{9}$$

When the swirl component u_0^θ of the initial data vanishes, (4) has a global unique regular solution [6–8]. However, if $u_0^\theta \neq 0$, then it is open for its global regularity as the full Navier–Stokes system. And thus, tremendous efforts and interesting progresses have been made on the regularity problem, see [9–19] and references cited therein. Now let us recall some regularity criteria which are relevant to our results. By [20, Theorem 1.1, Remark 1.3], we have the following two regularity conditions:

$$r^d u^\theta \in L^\alpha(0, T; L^\beta(\mathbb{R}^3)), \ \frac{2}{\alpha} + \frac{3}{\beta} = 1 - d, \ \frac{3}{1-d} < \beta \leqslant \infty, \ 0 \leqslant d < 1; \tag{10}$$

$$r^d u^\theta \in L^\alpha(0, T; L^\beta(\mathbb{R}^3)), \ \frac{2}{\alpha} + \frac{3}{\beta} = 1 - d, \ \frac{3}{1-d} < \beta \leqslant \infty, \ -1 \leqslant d < 0, \tag{11}$$

if $ru_0^\theta \in L^\infty$. Under this same condition on the initial data, we have the following three regularity criteria (see [20, Theorem 1.4, Corollary 1.5, Theorem 1.3] respectively)

$$r^d u^z \in L^\alpha(0, T; L^\beta(\mathbb{R}^3)), \ \frac{2}{\alpha} + \frac{3}{\beta} = 1 - d, \ \frac{3}{1-d} < \beta \leqslant \infty, \ 0 \leqslant d < 1, \tag{12}$$

$$ru^z \in L^\infty(0, T; L^\infty(\mathbb{R}^3)). \tag{13}$$