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A priori bounds for positive solutions of Kirchhoff type equations☆

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ABSTRACT

Let Ω be a bounded smooth domain in R^N. Assume that $0 < \alpha < \frac{2^*-1}{2}$, $a > 0$, and $b > 0$. We consider the following Dirichlet problem of Kirchhoff type equation

$$\begin{cases} -(a + b\,\|\nabla u\|_2^{2\alpha})\Delta u = |u|^{p-1}u + h(x, u, \nabla u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{0.1}$$

with $p \in (0,\ 2^*)\setminus\{1\}$. Where $2^* = +\infty$ for $N = 2$, and $2^* = \frac{N+2}{N-2}$ for $N \geq 3$. Under suitable conditions of $h(x, u, \nabla u)$ (see (A), (H_1) and (H_2) in Section 3), we get a priori estimates for positive solutions to problem (0.1). By making use of these estimates and the continuous method, we further get some existence results for positive solutions to problem (0.1) when $0 < p < 1$, or $2\alpha + 1 < p < 2^*$. Effects of the term $a + b\,\|\nabla u\|_2^{2\alpha}$ on the solution set of problem (0.1) can be seen in an example given in Section 2.

1. Introduction

Let $\Omega \subset R^N (N \geq 2)$ be an open bounded smooth domain with boundary $\partial\Omega$, $\|\bullet\|_q$ denotes the norm of $L^q(\Omega)$ for any $q > 0$. Assume that $a > 0$, $b > 0$ and $0 < \alpha < \frac{2^*-1}{2}$ with $2^* = +\infty$ for $N = 2$, and $2^* = \frac{N+2}{N-2}$ for $N \geq 3$. We consider the following problem for Kirchhoff type equations

$$\begin{cases} -(a + b\|\nabla u\|_2^{2\alpha})\Delta u = |u|^{p-1}u + h(x, u, \nabla u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

When $\alpha = 1$ and $h(x, u, \nabla u) = f(x, u)$, problem (1.1) is reduced to the following problem involving standard Kirchhoff operator

$$\begin{cases} -(a + b\|\nabla u\|_2^{2})\Delta u = |u|^{p-1}u + f(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

In recent years, problems like (1.2) have been extensively studied by making use of variational method. Mention some but few, we refer to [1–26] and the references cited therein.

In general, problem (1.1) has no variational structure. Hence, we adopt a device of using a priori estimate and fixed point theorem to study the positive solutions to it. This device has its benefits for it can ensure compactness of the positive

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solution set of problem (1.1) [27–30]. To make this device work, a crucial step is to drive an a priori estimate for solutions to the problem under consideration. It is believed that the existence result can be ensured once a priori estimates of solutions are established. However, this confidence is violated in our model problem (1.1). In fact, for arbitrary $a, b > 0$, we can get easily upper and lower bounds for nontrivial solutions to problem (1.1) when $1 < p < 2\alpha + 1$ and $h(x, u, \nabla u) \equiv 0$. Whereas, we find that problem (1.1) has nontrivial solution only for some a and b in this case (see Theorem 2.1(ii) in Section 2).

The rest of this paper is arranged as the following. Section 2 is used to analyze the solvability of problem (1.1) when $h(x, u, \nabla u) \equiv 0$. Section 3 devotes to derive a priori estimates of positive solutions to problem (1.1). Existence results for positive solutions to problem (1.1) with $0 < p < 1$, or $2\alpha + 1 < p < 2^*$ are given in Section 4.

2. A simple observation for $h(x, u, \nabla u) \equiv 0$

Assume that $p \in (0,\ 2^*) \setminus \{1\}$ and $0 < \alpha < \frac{2^*-1}{2}$. In this section, we analyze the solvability of the following problem

$$\begin{cases} -(a + b\|\nabla u\|_2^{2\alpha})\Delta u = |u|^{p-1}u, & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases} \tag{2.1}$$

It is easy to see that if u is a nontrivial solution of problem (2.1) and $v = \eta u$ with $\eta = (a + b\|\nabla u\|_2^{2\alpha})^{\frac{1}{1-p}}$, then v is a nontrivial solution of the following well studied problem

$$\begin{cases} -\Delta v = |v|^{p-1}v, & x \in \Omega, \\ v = 0, & x \in \partial\Omega. \end{cases} \tag{2.2}$$

Hence, any nontrivial solution of problem (2.1) can be obtained as the form $u = (a + b\beta^\alpha)^{\frac{1}{p-1}} v$ with v being a nontrivial solution of problem (2.2) and β being a positive solution of the following algebraic equation

$$y^{\frac{p-1}{2}} - b\|\nabla v\|_2^{p-1} y^\alpha - a\|\nabla v\|_2^{p-1} = 0.$$

Let

$$S(\Omega) = \inf_{0 \not\equiv u \in H_0^1(\Omega)} \frac{\|\nabla u\|_2^2}{\|u\|_{p+1}^2}.$$

Since $0 < p < 2^*$, it is well known that $S(\Omega)$ can be attained by some positive function $v \in H_0^1(\Omega)$. Let v be a solution of problem (2.2) which attains $S(\Omega)$. Then, it is easy to check that

$$\|\nabla v\|_2^{p-1} = S^{\frac{p+1}{2}}(\Omega).$$

For the simplicity of the notations, we set $S = S^{\frac{p+1}{2}}(\Omega)$ and $\gamma = 2\alpha + 1 - p$. Let

$$f(y) = y^{\frac{p-1}{2}} - bSy^\alpha - aS.$$

For $y > 0$, it is easy to see that

$$\begin{cases} f'(y) = y^{\alpha-1}(\dfrac{p-1}{2} y^{-\frac{\gamma}{2}} - \alpha bS), \\ f''(y) = y^{\alpha-2}(\dfrac{(p-1)(p-3)}{4} y^{-\frac{\gamma}{2}} - \alpha(\alpha-1)bS). \end{cases} \tag{2.3}$$

Therefore, we have

(i) If $0 < p < 1$, then $f'(y) < 0$ for $y \in (0,\ +\infty)$, $\lim_{y\to 0} f(y) = +\infty$, and $\lim_{y\to+\infty} f(y) = -\infty$. Hence, the equation $f(y) = 0$ has a unique solution in $(0,\ +\infty)$.

(ii) If $1 < p < 2\alpha + 1$, then $\gamma > 0$, and the equation $f'(y) = 0$ has a unique solution $y_0 = (\frac{p-1}{2\alpha bS})^{\frac{2}{\gamma}}$ in $(0,\ +\infty)$. Moreover,

$$\begin{cases} f'(y) > 0, & \text{for } y < y_0, \\ f'(y) < 0, & \text{for } y > y_0, \\ \max\limits_{y\in(0,\ +\infty)} \{f(y)\} = \gamma[\dfrac{(p-1)^{(p-1)}}{(2\alpha)^{(2\alpha)}}]^{\frac{1}{\gamma}} (bS)^{\frac{1-p}{\gamma}} - aS. \end{cases} \tag{2.4}$$

From this, we get

$$\begin{cases} \max\limits_{y\in(0,\ +\infty)} \{f(y)\} > 0, & \text{for } ab^{\frac{p-1}{\gamma}} < \gamma[\dfrac{(p-1)^{(p-1)}}{(2\alpha S)^{(2\alpha)}}]^{\frac{1}{\gamma}}, \\ \max\limits_{y\in(0,\ +\infty)} \{f(y)\} = 0, & \text{for } ab^{\frac{p-1}{\gamma}} = \gamma[\dfrac{(p-1)^{(p-1)}}{(2\alpha S)^{(2\alpha)}}]^{\frac{1}{\gamma}}, \\ \max\limits_{y\in(0,\ +\infty)} \{f(y)\} < 0, & \text{for } ab^{\frac{p-1}{\gamma}} > \gamma[\dfrac{(p-1)^{(p-1)}}{(2\alpha S)^{(2\alpha)}}]^{\frac{1}{\gamma}}. \end{cases} \tag{2.5}$$