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A force-based mathematical programming method for the incremental analysis of 3D frames with non-holonomic hardening plastic hinges

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ABSTRACT

In this work, a fully automated, step-by-step, 1st order matrix force method is developed for the analysis of planar (2D) and spatial (3D) structural frames made of elastic-hardening plastic material. Plasticity (ideal and hardening) is modelled using plastic hinges of zero-length. The proposed automation techniques utilize Lagrange multipliers to model all forms of discontinuities in a simple and efficient way, within the framework of mathematical programming: internal discontinuities (*e.g.* articulations) as well as element eccentricities are taken into account. The problem may be solved using any quadratic/non-linear optimization algorithm with linear constraints; the redundant forces/moments serve as the primary unknowns. A load-controlled numerical strategy that is suitable for any predefined loading scenario analysis is proposed; its efficiency is demonstrated via a set of examples which are compared with existing results from the literature or output from commercial software based on the equivalent direct stiffness method.

1. Introduction

The determination of a structural frame's maximum load bearing capacity and ductility without exceeding stability limits is of great interest to the practicing engineer: on one hand, it can be verified if an efficient design has been achieved by applying the traditional design/calculation procedures, and on the other hand, if a repair/strengthening is required in order for an existing structure to meet the new increased earthquake resistance regulatory standards. To this extend, a step-by-step, nonlinear procedure that uses plastic hinges may be applied (*e.g. pushover analysis*). Traditionally, such step-by-step procedures are formulated and carried out with the aid of the direct stiffness method.

Depending on whether holonomic or non-holonomic material behaviour is assumed, the determined plasticization event history by a step-by-step method may vary; in fact, in some cases, the assumption of holonomic behaviour has a negative effect on the proper determination of the collapse load and mechanism, because the phenomenon of local unstressing (*stress redistribution*) is neglected. Also, in the direct stiffness method, a predictor–corrector strategy is followed that requires a return to the yield surface; however, an implementation of a step-by-step analysis procedure that is based on mathematical programming requires no such return, because optimization algorithms function with a strict satisfaction of the constraint bounds. It is also well known that non-holonomic, post-elastic material behaviour is efficiently determined with the help of mathematical programming-based techniques. However, the displacement-based implementation of these processes contains a large number of unknowns. On the other hand, it is a known fact that the force method has certain computational advantages over the displacement method, such as for example, the lesser number of unknowns.

In the following, a chronological literature review in the fields of the matrix force method for structural frames and the applications of mathematical programming algorithms in plasticity is presented, with focus on structural engineering problems that may be modelled using beam/column finite elements with hardening plastic hinges of zero-length.

The first paper on the finite element method was published in 1954 [1]; in this work, a purely force-based approach to matrix structural analysis is also included. In 1956, two more papers were published, that signalled the development of the automation of the matrix force method for structural analysis [2,3]. In 1965, a complete theory for 3D elastic structural frames that follows a force-based approach in order to derive the governing equations was published [4]; in this work, the concept of "mutualities", which is

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a generalization of the Betti-Maxwell theorem of reciprocity, is also derived. A paper that proposes the usage of algebraic topological constants in the structural analysis of frames appeared in [5]. Formulations for plastic analysis and minimal weight design first appeared in [6]. The integrated force method was introduced in [7]. Some techniques for automatically selecting a cycle basis were proposed in [8]. Improvements in the algebraic procedures regarding the computation of the equilibrium matrices were discussed in [9]. Two schemata for generating the compatibility matrix in the integrated force method were presented in [10]. The first work that follows a force-based approach to elastoplastic analysis within the framework of mathematical programming was published in [11]; both static and kinematic approaches were formulated and solved using the dual algorithm of Lemke. An excellent review paper containing all up to date findings concerning the various approaches to the automation of the force method appeared in [12]. A direct flexibility method that is designed to work in interconnection with the direct stiffness method was proposed in [13]. Within the context of automated optimal plastic design, a novel automation technique was proposed in [14]. Advances of the integrated method were discussed in [15]. A fully automated method for optimal plastic design that is based on the upper bound (safe) theorem was presented in [16]. An extension of the integrated method to cater for non-linear structures was presented in [17]. A useful for educational purposes textbook on graph and matrix methods may be found in [18]; it also contains a detailed chapter on the force method. The incorporation of semi-rigid joints was introduced in [19].

The first papers where the correlation between structural analysis and Quadratic Programming (QP) is established, were authored in 1968 [20,21]. The QP approach concept was followed for formulating incremental elastoplastic analysis problems for elastic-work-hardening continua [22]; particular focus is given to the elastic-perfectly plastic material constitutive law. In [23], a set of Piece-Wise Linear (PWL) constitutive laws with associated plastic flow rules in matrix notation for the elastoplastic analysis of structures including kinematic and isotropic hardening is formulated; both incremental/step-by-step and shakedown analysis methods are addressed. Three procedures for the inelastic analysis of concrete beams under the assumption of material hardening behaviour with limited rotation capacity were proposed in [24]. The problem of incremental elastoplastic analysis of frames under hardening plastic hinges as a Parametric Linear Complementarity Problem (PLCP) was formed in [25]. In [26], a linearization to non linear yield criteria is applied, and in [27] a reformulation of [24,25] is devised and applied also to reinforced concrete frames. In [28], it was shown that a series of LP, Restricted Basis Linear Programming (RBLP), and QP problems may be formulated for evaluating the post elastic response of structures using a finite element approach under assumptions for elastic-perfectly plastic as well as hardening plastic material; in particular, for elastic-perfectly plastic material behaviour, the QP algorithms were found to yield the best performance. In [29], a solution schema for the PLCP is devised by using the Wolfe-Markowitz algorithm and by enforcing complementarity at each pivotal step of the Simplex method; the latter functions as a restriction of the variables that enter the basis. Slight modifications of the algorithm were sufficient to convert it to work for non-holonomic plasticity as well; extension to piecewise proportional loading is also demonstrated. In [30], an altered Linear Complementarity Problem (LCP) formulation for material hardening is proposed, that requires less computational time and storage. A RBLP technique [31] was applied in elastoplastic and limit analysis. In [32], a study on incremental elastoplastic constitutive laws that are primarily suitable for geotechnical and rocky materials but also fit for concrete was published; in this study, the dependence of the elastic moduli on plastic deformations is identified, as well as material instability phenomena within the range of material plastic hardening.

In [33], non-holonomic plasticity is incorporated with no particular remedies such as the ones required in [28,29], by working directly on incremental quantities. In [34], general purpose, step-by-step analysis computational procedures are provided, for both holonomic and non-holonomic material behaviour; they are based on a PLCP algorithm created by the authors. In [35], the significance of taking into account the effect of shear force in the evaluation of the limit load of frames is demonstrated; a formulation for limit analysis was devised, wherein the shear force effect is incorporated in the form of a parameter that affects the bending moment vs. axial force interaction surface; the DIN-18880 interaction surface for steel objects is also used as a comparative reference. A holonomic approach to the inverse problem of tracing the elastoplastic response of elastic-perfectly plastic and/or elastic–plastic hardening structures via a series of QP formulations which are solved by devising various numerical procedures that take into advantage the features of each formulation is proposed in [36]; these techniques have computational advantages as compared to other, generic 'direct search' optimization techniques. A state-of-the-art literature review paper on the applications of MP in engineering plasticity (*up to the year 1982*) was presented in [37]: subjects of plastic constitutive laws, plastic limit analysis, shakedown analysis, bounding methods, elastoplastic analysis, large displacement and instability analysis, contact problems and locking systems as well as dynamic plasticity are included; computational and theoretical aspects are also discussed. In an update to a previous literature review paper [37], the recent contributions in the field of MP applications to engineering plastic analysis up to the year 1986 are summarized in [38]; although the Interior Point Method (IPM) had already made its debut at the time, it was too early to have an impact on the aforementioned field, as commented by the authors. In [39], a MP-based approach to the elastoplastic analysis of frames was presented; although this procedure requires a large number of constraints –primarily due to the linearization of the non-linear convex yield surfaces used therein– it can efficiently trace phenomena of local unloading and the respective non-unique displacement fields.

In [40], a basis for formulating and solving non-linear optimization problems was devised using the parametric variational principles of minimum potential energy and minimum complementarity energy and by assuming that each yield function is expressed as the sum of two parts (*one function that is defined using the limits of the local stress components, and one function that is defined using the yield limit constants with reference to the constitutive law of the material*); the yield limits can be a generally non-linear functional of the internal variables/parameters that describe the material's constitutive law. The presented paper provides a unified framework for a variety of problems; it includes, but is not limited to, elastic-perfectly plastic and linear plastic hardening materials. In [41], based on the work presented in [33], the problem of non-uniqueness in the displacements computed in incremental non-holonomic elastoplastic analysis is investigated; with reference to the LCP, remedies regarding the reduction of memory storage and propagation of round-off errors for large-scale non-holonomic elastoplastic analysis were proposed in [42]. In [43], a method for the incremental analysis of plastic hardening structures that is cast on the basis of a Non-Linear Complementarity Problem (NLCP) was presented and tested for cyclic loading scenarios with good results.

A unified methodology for the inelastic analysis of skeletal structures with reference to the recent research findings on complementarity theory of MP is derived in [44]: A classical plastic hinge approach that obeys PWL material constitutive laws is followed, including both material plastic hardening and softening; both holonomic and non-holonomic approaches are discussed within the context of LCP-based formulations, for both rate-