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Eigenvalues of rotations and braids in spherical fusion categories

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ABSTRACT

We give formulae for the multiplicities of eigenvalues of generalized rotation operators in terms of generalized Frobenius–Schur indicators in a semisimple spherical tensor category $\mathcal{C}$. In particular, this implies that the entire collection of rotation eigenvalues for a fusion category can be computed from the fusion rules and the traces of rotation at finitely many tensor powers. We also establish a rigidity property for FS indicators of fusion categories with a given fusion ring via Jones's theory of planar algebras. If $\mathcal{C}$ is also braided, these formulae yield the multiplicities of eigenvalues for a large class of braids in the associated braid group representations. When $\mathcal{C}$ is modular, this allows one to determine the eigenvalues and multiplicities of braids in terms of just the S and T matrices.

1. Introduction

Our understanding of symmetry in a diverse range of topology, representation theory, and mathematical physics relies on first understanding *rotations*. Obvious examples appear in the study of winding numbers and knot polynomials. The actions of rotation operators played an important role in the construction and obstruction theory of

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Vaughan Jones's *planar algebras* [13], which provide a diagrammatic axiomatization of the representation theory of Murray–von Neumann subfactors. In particular, the small index subfactor classification program has been advanced significantly by understanding rotation eigenvalues in relation to other data; for an overview, see [14], [12].

From a purely algebraic standpoint rotation can be realized as a tensor flip map:

$$v_1 \otimes v_2 \otimes \cdots \otimes v_n \mapsto v_n \otimes v_1 \otimes v_2 \otimes \cdots \otimes v_{n-1}$$

Kashina, Sommerhäuser, and Zhu [18] showed that the traces of rotation operators on tensor powers of a given representation of a semisimple Hopf algebra yield the *higher Frobenius–Schur (FS) indicators* for the representation. (The definition of these in the Hopf algebra settings generalizes the classical Frobenius–Schur indicators defined for complex representations of finite groups; see [19].)

Ng and Schauenberg used a synthesis of the subfactor and Hopf algebra approaches to generalize much of this theory to the setting of *spherical fusion categories* [23][22]. The advantages of this setting are twofold. First, generality is sufficient to unify the representation theories for quantum groups [26], subfactors [20], conformal nets, and vertex operator algebras [17] along with the classification of fully extended 3-dimensional topological field theories [4]. Second, restrictions are sufficient to allow extensive use of graphical methods for morphisms.

The categorical FS indicators have proven to be a powerful invariant of fusion categories as well as their braided counterparts. They provide the main technical ingredient in the recent proofs showing the representation of the modular group $SL_2(\mathbb{Z})$ coming from the Drinfel'd center of a spherical fusion category (that is, the canonically associated braided fusion category) has a congruence subgroup as its kernel [28][24] and showing rank finiteness for *modular* categories [3]. Crucially, Ng and Schauenburg provide a method for computing FS indicators of a spherical fusion category in terms of the modular data of its Drinfel'd center [22]. The preceding is discussed in greater detail in section 2.

We have seen that there is a great deal of theory regarding the *traces* of rotation operators and how to compute them from basic categorical data, however their actual eigenvalues (and corresponding multiplicities) have been relatively inaccessible except in special cases. In this note we will give formulae for the multiplicities of the eigenvalues of (generalized) rotation operators in terms of the (generalized) FS indicators, and we will present two applications.

In section 3 we establish the eigenvalue multiplicities using Galois actions on cyclotomic fields and properties of (generalized) FS indicators to derive the traces of the powers of rotations. The finite Fourier transform yields the desired formulae in Theorem 3.4. However, it is desirable to be able to compute the rotation eigenvalues from as little information as possible since there are many situations where the full modular data for the center is not easily accessible. For example, in [9], Gannon and Morrison describe a procedure for computing possible modular data for the Drinfel'd centers $Z(\mathcal{C})$