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Discrete Morse theory and localization

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A B S T R A C T

Incidence relations among the cells of a regular CW complex produce a poset-enriched category of *entrance paths* whose classifying space is homotopy-equivalent to that complex. We show here that each acyclic partial matching (in the sense of discrete Morse theory) of the cells corresponds precisely to a homotopy-preserving localization of the associated entrance path category. Restricting attention further to the full localized subcategory spanned by critical cells, we obtain the *discrete flow category* whose classifying space is also shown to lie in the homotopy class of the original CW complex. This flow category forms a combinatorial and computable counterpart to the one described by Cohen, Jones and Segal in the context of smooth Morse theory.

1. Introduction

To the reader who desires a quick summary of this work, we recommend a brief glance at Fig. 1. Illustrated there is a regular CW complex $\mathbb{X}$ along with a simple operation which removes two cells x and y, where y is a face of x (written $x > y$).

Essentially, one drags y across x onto the other cells in the boundary of x. If y happens to be a *free face* of x (that is, if x is the unique cell of $\mathbb{X}$ containing the closure of y in its boundary), then our operation is an *elementary collapse* in the sense of simple homotopy theory [41]. In this special case, it is well-known that one can excise both x and y from $\mathbb{X}$ while preserving both homotopy type and regularity [8,18]. However, it is clear from our figure that if y is *not* a free face of x, then we must concoct a mechanism to glue other co-faces of y (such as x_0) to the remaining faces of x in order to preserve homotopy type once x and y have been removed. Our focus here is on providing an explicit and computable method to perform such attachments.

We immediately sacrifice regularity when pursuing these non-elementary collapses, and therefore must pay careful attention to how the remaining cells are attached. For instance, consider Fig. 1 again and note that $\mathbb{X}$ remains regular even if the vertices z_0 and z_1 are identified. However, our final complex in this case is not regular precisely because the cell x_0 becomes attached to this identified vertex $z_0 \sim z_1$ in

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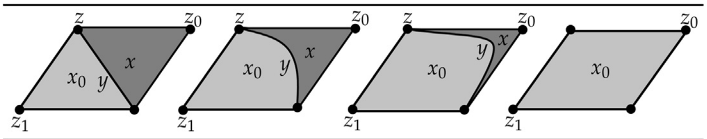

Fig. 1. Collapse of a cell pair (x, y) in the CW complex $\mathbb{X}$.

two essentially distinct ways.[1] In order to keep track of such alterations in attaching maps when several collapses are performed, we turn to the *entrance paths* [40] of $\mathbb{X}$. An entrance path from one cell to another is simply a descending sequence of intermediate faces connecting source to target — for instance, $(x > z)$ and $(x > y > z)$ are both entrance paths of $\mathbb{X}$ from x to z. The **entrance path category** of $\mathbb{X}$ is a poset-enriched category whose objects are the cells of $\mathbb{X}$, and whose morphisms are entrance paths partially ordered by inclusion, e.g., $(x > z) \Rightarrow (x > y > z)$. Finite regular CW complexes are homotopy equivalent to the classifying spaces of their entrance path categories.[2]

We show here that collapsing the cell pair x and y in $\mathbb{X}$ as described above corresponds to the **localization**, or formal inversion, of the morphism $(x > y)$ in the entrance path category of $\mathbb{X}$. The classifying space of the localized category so obtained is homotopy-equivalent to that of the un-localized one (and hence to $\mathbb{X}$). Moreover, one can safely remove both x and y from the localized category while preserving its homotopy type.

Motivation and related work. The central purpose of our work is to construct a Morse theory tailored to a class of poset-enriched categories broad enough to contain entrance path categories of all finite regular CW complexes. Aside from the natural desire to simplify computation of cellular homotopy (and weaker algebraic-topological invariants) by eliminating superfluous cells as described above, we are also motivated by at least two largely disjoint streams of existing results in Morse theory [29].

Forman's *discrete Morse theory* [14] has been successfully used to perform (co)homology computations not only in algebraic topology [11,30,31], but also in commutative algebra [21], topological combinatorics [37], algebraic combinatorics [35] and even geometric group theory [4]. The central idea involves the imposition of a *partial matching* μ on adjacent cell pairs of a regular CW complex $\mathbb{X}$ subject to a global acyclicity condition — the unmatched cells play the role of critical points whereas the matched cells generate combinatorial gradient-like flow paths. Although it is established that $\mathbb{X}$ is homotopy-equivalent to a CW complex whose cells correspond (in both number and dimension) to the critical cells of μ, there is no explicit description of how these critical cells are attached to each other. A second goal of this paper is to better understand the attaching maps in discrete Morse theory.

On the other hand, the relationship between *Morse theory and classifying spaces* in the smooth category has been described by Cohen, Jones and Segal in [9]. From a compact Riemannian manifold $\boldsymbol{X}$ equipped with a (smooth) Morse function $f : \boldsymbol{X} \to \mathbb{R}$, their work constructs a topologically enriched *flow category* $\boldsymbol{C}_f$ whose

- objects correspond to the critical points of f,
- morphisms are moduli spaces of broken gradient flow lines, and
- classifying space is homotopy-equivalent to $\boldsymbol{X}$.

Our third goal, then, is to produce a combinatorial and computable analogue of the flow category from Cohen–Jones–Segal's Morse theory by replacing Riemannian manifolds and smooth Morse functions by regular CW complexes and acyclic partial matchings.

[1] On the other hand, the attachment of x_0 to z remains unaltered across the collapse.
[2] See Proposition 3.3.

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