ARTICLE IN PRESS
JPAA:5888

Journal of Pure and Applied Algebra ••• (••••) •••–•••

Contents lists available at ScienceDirect

Journal of Pure and Applied Algebra

www.elsevier.com/locate/jpaa

Rigid realizations of modular forms in Calabi–Yau threefolds

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A R T I C L E I N F O

Article history:
Received 10 May 2017
Received in revised form 14 January 2018
Available online xxxx
Communicated by C. Hacon

MSC:
Primary: 14J32; secondary: 11F03; 14J15

A B S T R A C T

We construct examples of modular rigid Calabi–Yau threefolds, which give a realization of some new weight 4 cusp forms.

1. Introduction

The aim of this note is to construct new rigid Calabi–Yau realizations of weight 4 cusp forms as a resolution of a quotient of non-rigid examples listed by C. Mayer in [14].

Recall that a smooth, projective threefold is called a *Calabi–Yau* threefold if it satisfies the following two conditions:

(i) The canonical bundle of X is trivial and
(ii) $H^1(X, \mathcal{O}_X) = H^2(X, \mathcal{O}_X) = 0$.

A Calabi–Yau threefold is said to be *rigid* if it has no infinitesimal deformations i.e. $H^1(X, \mathcal{T}_X) = 0$. By Serre duality it is equivalent to the vanishing of $H^2(X, \Omega_X)$ or $h^{2,1}(X) = 0$.

One of the motivating problems in the study of the arithmetic of Calabi–Yau threefolds is the following conjecture:

Modularity conjecture. *Any rigid Calabi–Yau threefold X defined over $\mathbb{Q}$ is modular i.e. its L-series $L(X, s)$ coincides up to finitely many Euler factors with the Mellin transform $L(f, s)$ of a modular cusp form f of weight 4 with respect to $\Gamma_0(N)$, where the level N is only divisible by the primes of bad reduction.*

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https://doi.org/10.1016/j.jpaa.2018.04.005
0022-4049/

Please cite this article in press as: D. Burek, Rigid realizations of modular forms in Calabi–Yau threefolds, J. Pure Appl. Algebra (2018), https://doi.org/10.1016/j.jpaa.2018.04.005

In [7] and [8] L. Dieulefait and J. Manoharmayum proved the modularity conjecture for all rigid Calabi–Yau threefold defined over $\mathbb{Q}$ satisfying some mild assumptions on primes of bad reduction. Finally the conjecture was obtained in [9] from the proof of Serre's conjecture given by C. Khare and J.-P. Wintenberger; see [12] and [13].

B. Mazur and D. van Straten independently asked the opposite question if any weight 4 cusp form is a modular form of a rigid Calabi–Yau threefold. This question is still open, for a discussion of known examples see [15,16] and the references therein.

Acknowledgments. This paper is a part of author's master thesis. I am deeply grateful to my advisor Professor Sławomir Cynk for recommending me to learn this area and his help. We would like to thank anonymous referee whose comments improved the presentation of our paper.

2. Involutions of double octic Calabi–Yau threefolds

Let $S \subset \mathbb{P}^3$ be an arrangement of eight planes such that no six intersect and no four contain a line. Then there exists a resolution of singularities X of the double covering of $\mathbb{P}^3$ branched along S which is a Calabi–Yau threefold (see [6]).

A crepant resolution described in [6] is not uniquely determined but depends on the order of double lines. We can avoid this difficulty replacing the blow-up of all double lines separately with the blow-up of their sum in the singular double cover (see [5] or [2]). Now, by the universal property of a blow-up, every automorphism ϕ of the projective space $\mathbb{P}^3$ preserving S lifts to an automorphism $\widetilde{\phi}$ of the double octic X.

Any automorphism of a Calabi–Yau threefold acts on a canonical form ω of X by a multiplication with a scalar $\omega \mapsto \mu\omega$. If the automorphism has finite order n then $\mu^n = 1$. We shall call an automorphism *symplectic* if it preserves the canonical form ω i.e. $\mu = 1$.

We shall investigate in more details the case of an automorphism of order two. The fixed locus $\mathrm{Fix}(f)$ of a symplectic involution of a Calabi–Yau threefold is a disjoint union of smooth curves, while the fixed locus of a non-symplectic involution is a disjoint union of smooth surfaces and isolated points.

The branch locus S is a sum of eight planes, the multiplicity of a point P in S is the number of planes in S containing P.

2.1 Proposition. *Let ϕ be an automorphism of the projective space $\mathbb{P}^3$ of order two such that*

(i) *the fixed locus of ϕ contains no double nor triple line of S,*
(ii) *planes intersecting in fourfold point are not invariant by ϕ,*
(iii) *a fixed line of ϕ intersects S with at most two points of odd multiplicity.*

Then the fixed locus of the induced involution $\widetilde{\phi}\colon X \to X$ contains no curve with positive genus.

Proof. Our strategy is to perform the resolution and after each step verify that the fixed locus of the lifting of ϕ to the partial resolution contains no irrational curves. We start with the (singular) double covering of $\mathbb{P}^3$ branched along S. Since the fixed locus of an automorphism of $\mathbb{P}^3$ consists of two lines or a plane and point, we have two possibilities: either the image L of a fixed curve $\mathcal{C}$ of $\widetilde{\phi}$ is a fixed line of ϕ in $\mathbb{P}^3$ or it is an intersection line of a fixed plane of ϕ with an arrangement plane.

In the first case the inverse image of L in X is a double cover of L with at most two branch points of odd multiplicity, its normalization is a double cover of L with at most two single branch points, so it is a conic or a disjoint sum of two projective lines, which implies that $\mathcal{C}$ is itself rational. In the second case L is contained in the branch locus of the double covering, so $\mathcal{C}$ and L are isomorphic and hence rational curves.

The next step of the resolution is the blowing up of all fivefold points in the base of double covering. New branch divisor is the strict transform of S plus the exceptional divisors. So any fixed curve of the