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Frobenius base change of torsors

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A R T I C L E I N F O

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A B S T R A C T

We study the Frobenius base change of a torsor under a smooth algebraic group over a field of positive characteristic by relating it to the pushforward of the torsor under the Frobenius homomorphism. As an application, we determine the change of the multiplicity of a closed fiber of an elliptic surface by purely inseparable base changes with respect to the base curve in the case where the generic fiber is supersingular.

1. Introduction

We study the Frobenius base change of a torsor X under a smooth algebraic group G over a field K of positive characteristic p (i.e., G is a quasi-projective smooth K-group scheme). In the first part (§§2–4), we study the relationship between the Frobenius base change of X and the pushforward of X under the Frobenius homomorphism $G \to G^{(p)}$. In the last part (§§5–6), we apply the result in the first part to the case where X is the generic fiber of an elliptic fibration in order to determine the change of the multiplicity of a closed fiber of an elliptic surface by purely inseparable base changes with respect to the base curve.

Let us give details on the first part. Choose an algebraic closure K^{alg} of K. Take $n \in \mathbb{Z}_{\geq 0}$. Put $q := p^n$, $K_n := K^{\frac{1}{q}} := \{b \in K^{\mathrm{alg}} \mid b^q \in K\}$, and $S := \operatorname{Spec} K$. We denote the n-th iterate of the Frobenius homomorphisms by $F_{G/S,n} \colon G \to G^{(q)}$ (Definition 2.26). We define a K_n-group scheme G_n and a K_n-torsor X_n under G_n as the base changes of G and X via K_n/K, respectively, and a K-torsor $X^{(q)}$ under $G^{(q)}$ as the pushforward of X under $F_{G/S,n}$ (Definition 3.12). Recall that the first Galois cohomology $H^1(K, G)$ of K with coefficients in G may be regarded as the set of isomorphism classes of K-torsors under G (Definition 3.3 and Remark 3.13). We denote the cohomology class corresponding to the isomorphism class of X by $[X]$. We first construct a bijection $\phi^1_{G/S,n}$ such that the diagram

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$$\begin{array}{ccccc} & & \overset{b^1_{G/S,n}}{\curvearrowright} & & \\ H^1(K,G) & \xrightarrow[F^1_{G/S,n,*}]{} & H^1(K,G^{(q)}) & \xrightarrow[\phi^1_{G/S,n}]{\cong} & H^1(K_n,G_n) \end{array} \qquad (*)$$

commutes, where $b^1_{G/S,n}([X]) = [X_n]$ and $F^1_{G/S,n,*}([X]) = [X^{(q)}]$ for any K-torsor X under G (Definition 3.9 and Proposition 3.14). In the case $n = 1$, Diagram $(*)$ relates the Frobenius base change of X to the pushforward of X under the Frobenius homomorphism $G \to G^{(p)}$.

Assume that G is commutative. Then Diagram $(*)$ is a diagram of Abelian groups and homomorphisms (Remark 3.10). We denote the order of $[X_n]$ in $H^1(K_n, G_n)$ by m_n. As an application of Diagram $(*)$, we show the following behavior of $(m_n)_{n\geq 0}$ at the end of §4.

Theorem 1.1. *Assume that G is a superspecial K-Abelian variety, e.g., a supersingular K-elliptic curve* (*Definition* 4.1). *Then the following statements hold. If $p \mid m_n$, then one of the following equalities holds*:

(1) $(m_{n+1}, m_{n+2}) = (m_n/p, m_{n+1})$;
(2) $(m_{n+1}, m_{n+2}) = (m_n, m_{n+1}/p)$.

Otherwise, the equality $m_{n+1} = m_n$ holds.

In the proof of the above theorem, we decompose the multiplication of G by p into the two Frobenius homomorphisms and an isomorphism (Proposition 4.3), and apply Diagram $(*)$ for $n = 2$.

In the last part, we prove Theorem 1.2 below. Let $\pi: \mathcal{X} \to C$ be a relatively minimal elliptic fibration (Definition 5.3). The *multiplicity* of a closed fiber of π is defined as the greatest common divisor of the multiplicities of the irreducible components of the fiber. If the multiplicity is greater than one, then the fiber is called a *multiple fiber*. We consider the case where K is the function field of C, X is the generic fiber of $\mathcal{X}$, and G is the Jacobian of X. Take the normalization $u_n: C_n \to C$ of C in K_n. Let x be a closed point on C. The preimage $u_n^{-1}(x)$ consists of a single closed point on C_n since K_n is purely inseparable over K. We denote this closed point by x_n, the fiber over x_n of the minimal regular C_n-model of X_n by $\mathcal{X}_{x,n}$, and the multiplicity of the fiber $\mathcal{X}_{x,n}$ by $m_{x,n}$.

Theorem 1.2. *Let k be an algebraically closed field of positive characteristic p. Suppose that C is isomorphic to one of the following schemes*: (a) *a smooth k-curve*; (b) *the spectrum of the one-parameter formal power series ring with coefficients in k. Assume that G is supersingular. Then the following statements hold. If $p \mid m_{x,n}$, then one of the following equalities holds*:

(1) $(m_{x,n+1}, m_{x,n+2}) = (m_{x,n}/p, m_{x,n+1})$;
(2) $(m_{x,n+1}, m_{x,n+2}) = (m_{x,n}, m_{x,n+1}/p)$.

Otherwise, the equality $m_{x,n+1} = m_{x,n}$ holds.

We prove the above theorem at the end of §5. In the proof, we reduce the global case (a) to the local case (b) by base change with respect to C. In the local case, it is known that $m_{x,n} = m_n$ for any $n \in \mathbb{Z}_{\geq 0}$ (Theorem 5.7 (1)). Thus, Theorem 1.1 implies Theorem 1.2.

On the other hand, we may determine the type ${}_{m_{x,n}}T_n$ (Kodaira's symbol) of the fiber $\mathcal{X}_{x,n}$ in the following way. We denote the fiber over x_n of the minimal regular C_n-model of G_n by $\mathcal{G}_{x,n}$. Then T_n is equal to the type of $\mathcal{G}_{x,n}$ (Theorem 5.7 (1)), which may be determined from G_n by Tate's algorithm [23]. In

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