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The Frobenius complexity of Hibi rings

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ABSTRACT

We study the Frobenius complexity of Hibi rings over fields of characteristic $p > 0$. In particular, for a certain class of Hibi rings (which we call $\omega^{(-1)}$-level), we compute the limit of the Frobenius complexity as $p \to \infty$.

1. Introduction

Central to the study of singularities in characteristic p is the Frobenius morphism and its splittings. Given a commutative ring R of positive characteristic, the total Cartier algebra ($\mathcal{C}(R)$) is the graded, noncommutative ring of all potential Frobenius splittings of R, and it has been studied in various contexts in its relation to singularities [10], [14], [2]. Unfortunately, this ring is not finitely generated over R, even for relatively nice rings [1], [9], [8], but we can study the degree to which it is non-finitely generated. Specifically, Enescu and Yao defined a measure of the non-finite generation of this ring [4], called the Frobenius complexity ($cx_F(R)$), and they computed it for Segre products of polynomial rings. No other examples have been computed, and it is difficult to compute in a fixed characteristic. In particular, Enescu and Yao found that the Frobenius complexity of a Segre product is not even always rational. However, when they varied the base field and took a limit as $p \to \infty$, they found the limit Frobenius complexity is an integer in every case they studied. We will focus on computing the limit Frobenius complexity for a class of toric rings called Hibi rings, which are defined using finite posets. We will be able to compute it for Hibi rings which have a property we call $\omega^{(-1)}$-level, a condition related to the level condition which has been studied for Hibi rings in [12] and [13]. Our main theorem shows that the limit Frobenius complexity is an integer for $\omega^{(-1)}$-level Hibi rings, and in fact it can be read off directly from the poset. Specifically, we have the following.

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Main Theorem *(Theorem 4.10). If $R = \mathcal{R}_{\mathbb{F}_p}[\mathcal{I}(P)]$ is an $\omega_R^{(-1)}$-level (but non Gorenstein) Hibi ring associated to a poset P over $\mathbb{F}_p$, then*

$$\lim_{p\to\infty} cx_F(R) = \#\{\textit{elements of } P \textit{ which are not in a maximal chain of minimal length}\}.$$

Otherwise, in the Gorenstein case, we know $\mathcal{C}(R)$ is finitely generated over R, which means $cx_F(R) = -\infty$ [10]. As a particular case of this theorem, we recover the result of Enescu and Yao on the limit Frobenius complexity of Segre products of polynomial rings.

Frobenius complexity quantifies the minimal number of generators of $\mathcal{C}(R)_e$ for any e, which cannot be written as products of elements of lower degrees. We will give an upper bound on the number of generators of $\mathcal{C}(R)_e$ using the toric structure of Hibi rings. Then, we will use base p expansion techniques to give a lower bound by explicitly finding generators which are not products of elements of lower degrees. We will show these have the same order when our Hibi ring is $\omega^{(-1)}$-level.

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2. Background

2.1. Frobenius complexity

Let R be a ring of characteristic $p > 0$. Then for any R-module M, we can consider the set of p^{-e}-linear maps on M, namely all maps $\psi : M \to M$ such that

$$\psi(r^{p^e} m) = r\psi(m) \text{ and}$$
$$\psi(m_1 + m_2) = \psi(m_1) + \psi(m_2)$$

which we will denote $\mathcal{C}^e(M)$.

Similarly, we could consider the set of p^e-linear maps on M which we denote $\mathscr{F}^e(M)$. Namely, these are the maps $\phi : M \to M$ such that

$$\phi(rm) = r^{p^e}\phi(m) \text{ and}$$
$$\phi(m_1 + m_2) = \phi(m_1) + \phi(m_2)$$

Let $F^e : R \to R$ be the iterated Frobenius map, and let $F^e_* R$ denote the R-module which is isomorphic to R as a set (we write elements in $F^e_* R$ as $F^e_* r$ for some $r \in R$), but with an R-module structure given by:

$$r \cdot F^e_* x := F^e_*(r^{p^e} x) \text{ for all } r, x \in R$$

Similarly, for an R-module M, we let $F^e_* M$ be the R module which agrees with M as a set and has the multiplication structure $r \cdot F^e_* m = F^e_*(r^{p^e} m)$.

We can identify:

$$\mathcal{C}^e(M) \cong \operatorname{Hom}_R(F^e_* M, M)$$

and similarly

$$\mathscr{F}^e(M) \cong \operatorname{Hom}_R(M, F^e_* M)$$