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The Golod property for powers of ideals and Koszul ideals [☆]

Rasoul Ahangari Maleki [a,b,*]

[a] Department of Mathematics, Alzahra University, Zip Code 19834, Vanak, Tehran, Iran
[b] School of Mathematics, Institute for Research in Fundamental Sciences (IPM), P.O. Box: 19395-5746, Tehran, Iran

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ABSTRACT

Let S be a regular local ring or a polynomial ring over a field and I be an ideal of S. Motivated by a recent result of Herzog and Huneke, we study the natural question of whether I^m is a Golod ideal for all $m \geq 2$. We observe that the Golod property of an ideal can be detected through the vanishing of certain maps induced in homology. This observation leads us to generalize some known results from the graded case to local rings and obtain new classes of Golod ideals.

1. Introduction

Throughout this paper we let $(S, \mathfrak{n}, k)$ denote a regular local ring with maximal ideal $\mathfrak{n}$ and residue field k or a polynomial ring over a field k with graded maximal ideal $\mathfrak{n}$. All modules are assumed to be graded if the ring is graded. Let I be an ideal of S. The Poincaré series of a finitely generated $R = S/I$-module M is denoted by $P_M^R(t)$ and is defined to be formal power series $\sum_{i\geq 0} \dim_k \operatorname{Tor}_i^R(M,k)t^i$. In general, this power series is not a rational function. We refer the reader for the history of the rationality of Poincaré series to the survey article [1] by Avramov. On the other hand, Serre [27] showed that there is a coefficientwise inequality of formal power series

$$P_k^R(t) \leq \frac{(1+t)^d}{1 - t\sum_{i>0} \dim_k H_i(\mathcal{K})t^i}, \tag{1}$$

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* Correspondence to: Department of Mathematics, Alzahra University, Zip Code 19834, Vanak, Tehran, Iran.
E-mail address: rahangari@ipm.ir.

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where d is the embedding dimension of R and $\mathcal{K}$ is the Koszul complex of R with respect to a minimal system of generators of its maximal ideal.

We say that the ring R or the ideal I is Golod if $P_k^R(t)$ coincides with the upper bound given by Serre. Golod rings are an example of good rings in the sense that all finitely generated modules over such rings have rational Poincaré series sharing a common denominator, see [3]. Many results regarding classes of Golod ideals are established in the case of graded rings. If S is a polynomial ring over a field of characteristic zero, Herzog and Huneke in [13] identified large classes of Golod ideals. They showed, among other results, that the powers I^m of an ideal I are Golod for all $m \geq 2$.

The main goal of this paper is to study the Golod property of ideals of a regular local ring. In view of results of Herzog and Huneke, it is a natural question to ask whether the results of [13] hold if S is a polynomial ring over a field of arbitrary characteristic or more generally if S is a regular local ring rather than a polynomial ring. A known fact in this direction is a result of Herzog et al. [16] which says that large powers of an ideal are Golod. Another evidence in support of the question is that if I is a complete intersection, then I^m is Golod for all $m \geq 2$, see [2] and [9]. Herzog and the author [12] recently removed the assumption on characteristic of k of the results of [13] for monomial ideals. The methods used in the proofs of the results we mentioned above vary from one case to another, yet the nature of the results suggests that there is some common ground among them. This observation allows us to partially generalize to local rings some of the known results for graded rings.

Let $\rho(I)$ denote the smallest number m such that for all $r > m$ and all $i > 0$ the natural map $\operatorname{Tor}_i^S(S/I^r, k) \to \operatorname{Tor}_i^S(S/I^{r-1}, k)$ is the zero map. In Section 2 we show that the invariant $\rho(I)$ is finite and the following holds:

Theorem 1.1. *Let m be a positive integer and J be an ideal of S. Then the following hold:*

1. *If $m > \rho(I)$ and $I^{2m-2} \subseteq J \subseteq I^m$, then J is Golod. In particular, I^m is Golod.*
2. *If $d = \dim S$ and $m \geq \max\{2d, \rho(I) + d\}$. Then $\overline{I^m}$, the integral closure of I^m, is Golod.*

If $\rho(I) = 1$, then it follows at once from this theorem that I^m is Golod for all $m \geq 2$. This suggests the question

Question 1.2. Is it true that $\rho(I) = 1$ for any ideal I of S?

We show that Question 1.2 has an affirmative answer provided that either S is a polynomial ring over a field of characteristic zero or S has Krull dimension at most 2, see Proposition 3.5 and Theorem 3.7. The result in the graded case is an immediate consequence of the work of Herzog and Huneke. When S is local, we prove in Proposition 3.8 that $\rho(I) = 1$ if I is a complete intersection ideal. A similar conclusion for a monomial ideal I, without any assumption on characteristic, holds true, see [12].

Section 3 of this paper is devoted to study the Golod property of Koszul ideals. We say that an ideal I of S is Koszul if its associated graded module with respect to maximal ideal $\mathfrak{n}$ has a linear resolution over the associated graded ring $\oplus_{i\geq 0}\mathfrak{n}^i/\mathfrak{n}^{i+1}$. This notion can be considered as a generalization of the notion of componentwise linear ideal. A necessary (not sufficient) condition for an ideal I to be Koszul is that the natural map $\upsilon_i^S(I) : \operatorname{Tor}_i^S(\mathfrak{n}I, k) \to \operatorname{Tor}_i^S(I, k)$ is zero for all i. In [15] it is proved that any componentwise linear ideal of a polynomial ring S is Golod. We extend this result by showing that

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