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An example of an atomic pullback without the ACCP

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ABSTRACT

We find necessary and sufficient conditions on a pullback diagram in order that every nonzero nonunit in its pullback ring admits a finite factorization into irreducible elements. As a result, we can describe a method of easily producing atomic domains that do not satisfy the ascending chain condition on principal ideals.

1. Introduction

Let D be any integral domain with field of fractions K and let $D^\bullet$ denote the set of nonzero nonunit elements of D. An element $d \in D^\bullet$ is called *irreducible* (or an *atom*), if it cannot be written as a nontrivial product of elements from $D^\bullet$. We will write $\mathrm{Irr}(D)$ to denote the set of all irreducible elements of D. An element $d \in D^\bullet$ is called *atomic* if it admits a finite factorization $d = \pi_1\pi_2 \cdots \pi_t$ with $\pi_i \in \mathrm{Irr}(D)$ for each $i \leq t$. Let $\mathcal{F}(D)$ be the set of all atomic elements of D and $\mathcal{N}(D) = D^\bullet - \mathcal{F}(D)$. That is, $\mathcal{N}(D)$ is the set of elements of $D^\bullet$ that do not admit a factorization into irreducibles.

The domain D is called *atomic* if $\mathcal{N}(D) = \varnothing$. Some standard examples of atomic domains include UFDs (every factorization of α into irreducibles has the same length and is unique up to associates), HFDs (every factorization of α into irreducibles has the same length), Noetherian domains (ascending chain condition on ideals), and domains satisfying ACCP (ascending chain condition on principal ideals). It is well known that we have the chain of implications displayed below.

$$\text{UFD} \Rightarrow \text{HFD} \Rightarrow \text{ACCP} \Rightarrow \text{Atomic}$$

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The purpose of this article is to examine the behavior of atomicity in certain pullback constructions. In particular, we consider a special type of conductor square introduced in [2] that defines a ring between $D[x]$ and $K[x]$. Let $v(x) = v_1(x) \cdots v_r(x)$ where $v_1, ..., v_r$ are distinct irreducible polynomials over the field K. If $C = v(x)K[X]$, then we have the natural surjection

$$\eta : K[x] \twoheadrightarrow K[x]/C \simeq \Pi_{i=1}^{r} K[\theta_i]$$

where, for each index $i \leq r$, θ_i is a root of v_i. If D_i is any overring of $D[\theta_i]$, then we have the inclusion

$$\iota : \Pi_{i=1}^{r} D_i \hookrightarrow \Pi_{i=1}^{r} K[\theta_i].$$

Taking the pullback of the maps η and ι, we obtain the ring

$$R = \{g(x) \in K[x] : g(\theta_i) \in D_i \text{ for each } i \leq r\}$$

between $D[x]$ and $K[x]$ with the non-zero conductor C from $K[x]$ into R. We will say that R *is defined by a conductor square of the type* (⊠).

$$\begin{array}{ccc} R & \hookrightarrow & K[x] \\ \downarrow & & \downarrow \\ \Pi_{i=1}^{r} D_i & \hookrightarrow & \Pi_{i=1}^{r} K[\theta_i] \end{array} \qquad (\boxtimes)$$

Recall that if $E = \{e_1, ..., e_r\}$ is a subset of D, then $\text{Int}(E, D) = \{g \in K[x] : g(E) \in D\}$ is called the ring of integer-valued polynomials on D determined by E. It was first proved in [15] that $\text{Int}(E, D) = f(x)K[x] + \Sigma_{i=1}^{r} D\phi_i(x)$ where $f(x) = (x - e_1) \cdots (x - e_r)$ and, for each $i \leq r$, the polynomial ϕ_i is the ith LaGrange interpolation polynomial on the set E. This representation indicates that $\text{Int}(E, D)$ is definable by a conductor square of the type (⊠). Indeed, it is noted in [2] that if we set $v_i(x) = (x - e_i)$ and $D_i = D$ for each $i \leq r$, then the resulting pullback ring is $R = \text{Int}(E, D)$.

Much is known about the ring $\text{Int}(E, D)$ when E is finite. For example, [15] uses the representation above to show that $\text{Int}(E, D)$ is a Prüfer domain if and only if D is a Prüfer domain. Also, [8] proves that $\text{Int}(E, D)$ has the strong 2-generator property if and only if D is a Bézout domain. A similar result for $\text{Int}(E, D)$ can be found in [4] for a larger number of generators. As the previous paragraph suggests, analogous results hold for a ring R defined by a conductor square of the type (⊠).

Some authors have also considered factorization in various pullback constructions in the spirit of $\text{Int}(E, D)$. For example, [11] finds necessary and sufficient conditions on the pullback diagram defining $S = A + xB[x]$ in order that S is an HFD. On the other hand, [1] proves that if D is not a field, then $\text{Int}(E, D)$ is never atomic. Moreover, a ring R defined by the construction (⊠) does not satisfy ACCP (see below). At this point it would be reasonable to guess that the ring R is not atomic.

Indeed, P.M. Cohn states without proof in [10], that atomicity and ACCP are the same. However, in [12], Anne Grams proved that this claim is false using a 1-dimensional quasilocal atomic monoid domain that does not satisfy ACCP. In [17], Zaks added two more examples of an atomic domain without ACCP similar to the Grams example. Other atomic examples without the ACCP similar to Zaks' are exhibited in [16]. The examples in [12], [17], and [16] are the only ones we know of that are atomic but do not satisfy the ACCP. All of these examples are constructed from monoid domains or a large number of fractions adjoined to polynomial rings with several variables.

Since pullback constructions are known to be a rich source of counterexamples (see [13]), we have further motivation to check the atomicity in a ring R defined by the diagram (⊠) more carefully. The main result of this paper is to exhibit necessary and sufficient conditions on the polynomial $v(x)$, the conductor ideal $C = v(x)K[x]$, and the ring of constants of R in order to ensure that the ring R is atomic. In conclusion,

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