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Stillman's question for exterior algebras and Herzog's conjecture on Betti numbers of syzygy modules

Jason McCullough

Iowa State University, Department of Mathematics, Ames, IA 50011, United States of America

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ABSTRACT

Let K be a field of characteristic 0 and consider exterior algebras of finite dimensional K-vector spaces. In this short paper we exhibit principal quadric ideals in a family whose Castelnuovo–Mumford regularity is unbounded. This negatively answers the analogue of Stillman's Question for exterior algebras posed by I. Peeva. We show that, via the Bernstein–Gel'fand–Gel'fand correspondence, these examples also yields counterexamples to a conjecture of J. Herzog on the Betti numbers in the linear strand of syzygy modules over polynomial rings.
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1. Introduction

Let K be a field and let S be the symmetric algebra of some finite dimensional K-vector space. Stillman [15, Problem 3.14] posed the following question: Can the projective dimension $\mathrm{pd}_S(S/I)$ of a homogeneous ideal I be bounded purely in terms of the number and degrees of the minimal generators of I? Caviglia showed that this question was equivalent to the analogous question where one replaces projective dimension by regularity (cf. [13, Theorem 2.4]). Ananyan and Hochster [1] recently gave a positive answer to this question in full generality. More recent proofs have been given by Erman, Sam, and Snowden [7] and Draisma, Lason, and Leykin [3].

Now let E be the positively graded exterior algebra of a finite dimensional K-vector space. While resolutions over E need not be finite, the regularity $\mathrm{reg}_E(M)$ of a finitely generated E-module is finite since E is a Koszul algebra. Here $\mathrm{reg}_E(M)$ is defined as

$$\mathrm{reg}_E(M) := \sup\{j - i|\ \mathrm{Tor}_i^E(M, K)_j \neq 0\}.$$

Irena Peeva posed the following variant of Stillman's Question (cf. [12, p. 241]) at the Joint Introductory Workshop at MSRI in the Fall of 2012:

E-mail address: jmccullo@iastate.edu.

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Question 1.1 *(Peeva). Can the regularity* $\operatorname{reg}_E(E/I)$ *of a homogeneous ideal* I *of* E *be bounded purely in terms of the number and degrees of the generators?*

Surprisingly, and contrary to the symmetric algebra case, the answer to this question is no. In Section 4 we present a family of principal quadric ideals in exterior algebras over an arbitrary field whose regularity is unbounded.

Let M be a finitely generated, graded S-module, where $S = K[x_1, \ldots, x_n]$. Let d denote the minimal degree of a generator of M. We consider the Betti numbers in the linear strand of M, that is:

$$\beta_i^{\mathrm{lin}}(M) := \beta_{i,d+i}(M),$$

where $\beta_{i,j}(M) = \dim_K \operatorname{Tor}_i^S(M, K)_j$. The *length of the linear strand* of M is $\max\{i \mid \beta_i^{\mathrm{lin}}(M) \neq 0\}$. Herzog proposed the following lower bound on the Betti numbers in the linear strand of kth syzygy modules.

Conjecture 1.2 *(Herzog [10]). If* M *is a graded* k*th syzygy module over* S *with linear strand of length* p*, then*

$$\beta_i^{lin}(M) \geq \binom{p+k}{i+k}.$$

The conjecture has been proved in the following cases:

1. Herzog proved the case $k = 0$ in [10].
2. Herzog was motivated by a result of Green [9] which contained the case $k = 1, i = 0$. See also similar results by Eisenbud and Koh [6].
3. Reiner and Welker [16] proved the case when M is a monomial ideal and $k = 1$.
4. Römer [17] proved the following cases:
   (a) When $k = 1$ and $0 \leq i \leq p$.
   (b) When $p > 0$ and $i = p - 1$; i.e. $\beta_{p-1}^{\mathrm{lin}}(M) \geq \binom{p+k}{p-1+k} = p + k$.
   (c) When M is $\mathbb{Z}^n$-graded, Römer proved the conjecture in full generality.

By modifying a recent construction of Iyengar and Walker [11], we use the Bernstein–Gel'fand–Gel'fand (BGG) correspondence to produce counterexamples for virtually all other cases of Herzog's Conjecture. More precisely, in Section 3 we construct for each $n \geq 1$ a finitely generated, graded $S = K[x_1, \ldots, x_{2n}]$-module M that is an nth syzygy module such that M has linear resolution of length n and has graded Betti numbers

$$\beta_i^{\mathrm{lin}}(M) = \binom{2n}{n+i} - \binom{2n}{n+i+2}, \qquad \text{for } 0 \leq i \leq n.$$

We note that for $0 \leq i \leq n - 2$ that

$$\beta_i^{\mathrm{lin}}(M) < \binom{n+n}{n+i},$$

contradicting the conjecture. These examples are BGG dual to the principal quadric E-ideals mentioned above.

We note that a different construction of Conca, Herbig, and Iyengar [2] also gives a family of counterexamples to Conjecture 1.2, although that was not their aim. Specifically, they show [2, Theorem 5.1] that if $S = K[x_1, \ldots, x_n, y_1, \ldots, y_n]$ and I is the ideal

$$I = (\{x_i y_j, x_i y_j - x_j y_i \mid 1 \leq i, j \leq n \text{ with } i \neq j\}),$$

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