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On Hopf algebras over quantum subgroups ☆

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ABSTRACT

Using the standard filtration associated with a generalized lifting method, we determine all finite-dimensional Hopf algebras over an algebraically closed field of characteristic zero whose coradical generates a Hopf subalgebra isomorphic to the smallest non-pointed non-cosemisimple Hopf algebra $\mathcal{K}$ and the corresponding infinitesimal module is an indecomposable object in ${}^{\mathcal{K}}_{\mathcal{K}}\mathcal{YD}$ (we assume that the diagrams are Nichols algebras). As a byproduct, we obtain new Nichols algebras of dimension 8 and new Hopf algebras of dimension 64.

0. Introduction

Let $\Bbbk$ be an algebraically closed field of characteristic zero. The problem of classifying all Hopf algebras over $\Bbbk$ of a given dimension was posed by Kaplansky in 1975 [8]. Some progress has been made but, in general, it is a difficult question. One of the few general techniques is the so-called *Lifting Method* [3], under the assumption that the coradical is a subalgebra, *i.e.*, the Hopf algebra has the Chevalley Property. More recently, Andruskiewitsch and Cuadra [1] proposed to extend this technique by considering the subalgebra generated by the coradical and the related wedge filtration. It turns out that this filtration is a Hopf algebra filtration, provided that the antipode is injective, what is true in the finite-dimensional context.

We describe the lifting method briefly. Let H be a Hopf algebra over $\Bbbk$. Recall that the coradical filtration $\{H_n\}_{n\geq 0}$ of H is defined recursively by

- the coradical H_0, which is the sum of all simple subcoalgebras, and
- $H_n = \bigwedge^{n+1} H_0 = \{h \in H : \ \Delta(h) \in H \otimes H_0 + H_{n-1} \otimes H\}$.

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This filtration corresponds to the filtration of H^* given by the powers of the Jacobson radical. It is always a coalgebra filtration and if H_0 is a Hopf subalgebra, then it is indeed a Hopf algebra filtration; in particular, its associated graded object $\operatorname{gr} H = \bigoplus_{n\geq 0} H_n/H_{n-1}$ is a graded Hopf algebra, where $H_{-1} = 0$. Let $\pi : \operatorname{gr} H \to H_0$ be the homogeneous projection. It turns out that $\operatorname{gr} H \simeq R\#H_0$ as Hopf algebras, where $R = (\operatorname{gr} H)^{\operatorname{co}\pi} = \{h \in H : (\mathrm{id} \otimes\pi)\Delta(h) = h \otimes 1\}$ is the algebra of coinvariants and $\#$ stands for the Radford–Majid biproduct or *bosonization* of R with H_0. The algebra R is not a usual Hopf algebra, but a graded connected Hopf algebra in the category ${}^{H_0}_{H_0}\mathcal{YD}$ of left Yetter–Drinfeld modules over H_0. The subalgebra generated by the elements of degree one is the *Nichols algebra* $\mathfrak{B}(V)$ of $V = R(1)$; here V is a braided vector space called the *infinitesimal braiding*.

Let us fix a finite-dimensional cosemisimple Hopf algebra A. The lifting method then consists of the description of all finite-dimensional Nichols algebras $\mathfrak{B}(V) \in {}^{A}_{A}\mathcal{YD}$, the determination of all possible deformations of the bosonization $\mathfrak{B}(V)\#A$, and the proof that all Hopf algebras H with $H_0 = A$ satisfy that $\operatorname{gr} H \simeq \mathfrak{B}(V)\#A$.

The main idea in [1] is to replace the coradical filtration by a more general but adequate filtration: the **standard filtration** $\{H_{[n]}\}_{n\geq 0}$, which is defined recursively by

- the subalgebra $H_{[0]}$ of H generated by H_0, called the *Hopf coradical*, and
- $H_{[n]} = \bigwedge^{n+1} H_{[0]}$.

If the coradical H_0 is a Hopf subalgebra, then $H_{[0]} = H_0$ and the coradical filtration coincides with the standard one.

Let A be a Hopf algebra generated by its coradical. We will say that H is a *Hopf algebra over* A if $H_{[0]} \simeq A$ as Hopf algebras.

Assume that the antipode $\mathcal{S}$ of H is injective. Then by [1, Lemma 1.1], it holds that $H_{[0]}$ is a Hopf subalgebra of H, $H_n \subseteq H_{[n]}$ and $\{H_{[n]}\}_{n\geq 0}$ is a Hopf algebra filtration of H. In particular, the graded algebra $\operatorname{gr} H = \bigoplus_{n\geq 0} H_{[n]}/H_{[n-1]}$ with $H_{[-1]} = 0$ is a Hopf algebra associated with the standard filtration. Write $\pi : \operatorname{gr} H \to H_{[0]}$ for the homogeneous projection. Then, as before, it splits the inclusion of $H_{[0]}$ in $\operatorname{gr} H$, the *diagram* $R = (\operatorname{gr} H)^{\operatorname{co}\pi}$ is a Hopf algebra in the category ${}^{H_{[0]}}_{H_{[0]}}\mathcal{YD}$ of Yetter–Drinfeld modules over $H_{[0]}$ and $\operatorname{gr} H \simeq R\#H_{[0]}$ as Hopf algebras. It turns out that $R = \bigoplus_{n\geq 0} R(n)$ is also graded and connected. We call again the linear space $R(1)$ consisting of elements of degree one, the *infinitesimal braiding*.

The procedure to describe explicitly any Hopf algebra as above defines a proposal for the classification of general finite-dimensional Hopf algebras over a fixed Hopf subalgebra A which is generated by a cosemisimple coalgebra. The main steps are the following:

(a) determine all Yetter–Drinfeld modules V in ${}^{A}_{A}\mathcal{YD}$ such that the Nichols algebra $\mathfrak{B}(V)$ is finite-dimensional,
(b) for such V, compute all Hopf algebras L such that $\operatorname{gr} L \simeq \mathfrak{B}(V)\#A$. We call L a *lifting* of $\mathfrak{B}(V)$ over A,
(c) prove that any finite-dimensional Hopf algebra over A is generated by the first term of the standard filtration.

In this paper, we study these questions (a) and (b) in the case that $A = \mathcal{K}$ is the smallest Hopf algebra whose coradical is not a subalgebra. It is an 8-dimensional Hopf algebra whose dual is a pointed Hopf algebra. The dual Hopf algebra A^* was first introduced by Radford [11], who addressed the problem of finding a Hopf algebra whose Jacobson radical is not a Hopf ideal.

Let ξ be a primitive 4-th root of 1. As an algebra, $\mathcal{K}$ is generated by the elements a, b, c, d satisfying the following relations:

$$\begin{aligned} &ab = \xi ba, && ac = \xi ca, && 0 = cb = bc, && cd = \xi dc, && bd = \xi db, \\ &ad = da, && ad = 1, && 0 = b^2 = c^2, && a^2c = b, && a^4 = 1. \end{aligned} \tag{1}$$