# Accepted Manuscript


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Eugenio Giannelli, Gunter Malle, Carolina Vallejo Rodríguez

PII: S0022-4049(18)30127-0
DOI: https://doi.org/10.1016/j.jpaa.2018.05.004
Reference: JPAA 5908

To appear in: *Journal of Pure and Applied Algebra*

Received date: 25 October 2017
Revised date: 19 April 2018

Please cite this article in press as: E. Giannelli et al., Even degree characters in principal blocks, *J. Pure Appl. Algebra* (2018), https://doi.org/10.1016/j.jpaa.2018.05.004

# EVEN DEGREE CHARACTERS IN PRINCIPAL BLOCKS

EUGENIO GIANNELLI, GUNTER MALLE, AND CAROLINA VALLEJO RODRÍGUEZ

ABSTRACT. We characterise finite groups such that for an odd prime $p$ all the irreducible characters in its principal $p$-block have odd degree. We show that this situation does not occur in non-abelian simple groups of order divisible by $p$ unless $p = 7$ and the group is $M_{22}$. As a consequence we deduce that if $p \neq 7$ or if $M_{22}$ is not a composition factor of a group $G$, then the condition above is equivalent to $G/\mathbf{O}_{p'}(G)$ having odd order.

## 1. INTRODUCTION

Let $G$ be a finite group, let $p$ be a prime dividing the order of $G$ and let $B_0$ be the Brauer principal ($p$-)block of $G$. Brauer's height zero conjecture asserts that $p$ does not divide the degrees of the irreducible ordinary characters belonging to $B_0$ if, and only if, a Sylow $p$-subgroup $P$ of $G$ is abelian. Let $q$ be a prime different from $p$. It would be interesting to characterise when all degrees of irreducible ordinary characters belonging to $B_0$ are coprime with $q$. When $q = 2$, G. Navarro predicted that all irreducible ordinary characters in $B_0$ have odd degree if, and only if, $G/\mathbf{O}_{p'}(G)$ has odd order. We confirm here that this claim holds whenever $p \neq 7$. For $p = 7$, the group $M_{22}$ is a counterexample, and the only counterexample among finite simple groups.

**Theorem A.** *Let $p$ be an odd prime, and let $B_0$ be the principal $p$-block of a group $G$ of order divisible by $p$. If $p \neq 7$ or $M_{22}$ is not a composition factor of $G$, then every irreducible character in $B_0$ has odd degree if, and only if, $G/\mathbf{O}_{p'}(G)$ is a group of odd order.*

The Ito–Michler theorem characterises when a prime $q$ does not divide the degrees of the irreducible characters of a group. A natural version of the Ito–Michler theorem for principal blocks would be: If all the irreducible characters of $B_0(G)$ have degree coprime to $q$, for some prime $q \neq p$, then some Sylow $q$-subgroup $Q$ of $G$ is normalised by a Sylow $p$-subgroup $P$ of $G$. In [NW01] the authors prove this result under the assumption that $G$ is a $\{p, q\}$-separable group. However, such a version does not hold outside $\{p, q\}$-separable groups, as the authors also point out that the separability condition of $G$ is necessary (as shown by $G = J_1$, $p = 2$, $q = 5$). For $q = 2$ and $p \neq 7$ we have the characterisation given by Theorem A. (Observe that a Sylow 2-subgroup of $G = M_{22}$ is self-normalising and hence in particular not normalised by any Sylow 7-subgroup of $G$.)

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2010 *Mathematics Subject Classification.* 20C15, 20C20, 20C30, 20C33.

*Key words and phrases.* Principal blocks, even degree characters.

The first author's research is funded by Trinity Hall, University of Cambridge. The second author gratefully acknowledges financial support by SFB TRR 195. The third author is partially supported by the Spanish Ministerio de Educación y Ciencia Proyectos MTM2016-76196-P and the ICMAT Severo Ochoa project SEV-2011-0087.