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Full Length Article

Simultaneous approximation by Bernstein polynomials with integer coefficients

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Abstract

We prove that several forms of the Bernstein polynomials with integer coefficients possess the property of simultaneous approximation, that is, they approximate not only the function but also its derivatives. We establish direct estimates of the error of that approximation in uniform norm by means of moduli of smoothness. Moreover, we show that the sufficient conditions under which those estimates hold are also necessary.

Keywords: Bernstein polynomials; Integer coefficients; Integral coefficients; Simultaneous approximation; Rate of convergence; Modulus of smoothness

1. Main results

The Bernstein operator or polynomial is defined for $f \in C[0, 1]$ and $x \in [0, 1]$ by

$$B_n f(x) := \sum_{k=0}^{n} f\left(\frac{k}{n}\right) p_{n,k}(x), \qquad p_{n,k}(x) := \binom{n}{k} x^k (1-x)^{n-k}.$$

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Here $n \in \mathbb{N}$, where $\mathbb{N}$ is the set of the *positive* integers. It is known that if $f \in C[0,1]$, then

$$\lim_{n\to\infty} \|B_n f - f\| = 0,$$

where $\|\circ\|$ is the sup-norm on the interval $[0,1]$. A best possible estimate of that convergence can be given by the Ditzian–Totik modulus of smoothness $\omega_\varphi^2(f,t)$ of the second order with a varying step, controlled by the weight $\varphi(x) := \sqrt{x(1-x)}$, in the uniform norm on the interval $[0,1]$. It is defined by (see [4, Chapter 2, (2.1.2)])

$$\omega_\varphi^2(f,t) := \sup_{0<h\le t} \|\bar{\Delta}_{h\varphi}^2 f\|,$$

where

$$\bar{\Delta}_{h\varphi(x)}^2 f(x) := \begin{cases} f(x+h\varphi(x)) - 2f(x) + f(x-h\varphi(x)), & x \pm h\varphi(x) \in [0,1], \\ 0, & \text{otherwise.} \end{cases}$$

For all $f \in C[0,1]$ and $n \in \mathbb{N}$ there holds (see [3, Chapter 10, (7.3)], or [2, Theorem 6.1])

$$\|B_n f - f\| \le c\, \omega_\varphi^2(f, n^{-1/2}). \tag{1.1}$$

Above and henceforward c denotes a positive constant, not necessarily the same at each occurrence, whose value is independent of f and n.

The estimate (1.1) is best possible in the sense that its converse also holds true (see [12] and [20], or [3, Chapter 10, (7.3)], or [2, Theorem 6.1])

$$\omega_\varphi^2(f, n^{-1/2}) \le c\, \|B_n f - f\|.$$

The varying-step moduli are quite useful when the approximation is better near the endpoints of the interval. Such is the case of the Bernstein polynomials, which interpolate the function at 0 and 1. More importantly, these moduli (unlike the classical ones) allow better inverse theorems for the best algebraic approximation since they take into account the effect of the endpoints (see [4, Chapter 7] and [3, Chapter 8]). Instead of $\omega_\varphi^2(f,t)$ we can use the moduli defined in [9,10,7,13–15,19], or [6].

Kantorovich [11] (or e.g. [1, pp. 3–4], or [16, Chapter 2, Theorem 4.1]) introduced an integer modification of B_n. It is given by

$$\widetilde{B}_n(f)(x) := \sum_{k=0}^{n} \left[f\left(\frac{k}{n}\right) \binom{n}{k} \right] x^k (1-x)^{n-k}.$$

Above $[\alpha]$ denotes the largest integer that is less than or equal to the real α. L. Kantorovich showed that if $f \in C[0,1]$ is such that $f(0), f(1) \in \mathbb{Z}$, then

$$\lim_{n\to\infty} \|\widetilde{B}_n(f) - f\| = 0.$$

Clearly, the conditions $f(0), f(1) \in \mathbb{Z}$ are also necessary in order to have $\lim_{n\to\infty} \widetilde{B}_n(f)(0) = f(0)$ and $\lim_{n\to\infty} \widetilde{B}_n(f)(1) = f(1)$, respectively.