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Full Length Article

Recovery algorithms for high-dimensional rank one tensors

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Abstract

We present deterministic algorithms for the uniform recovery of d-variate rank one tensors from function values. These tensors are given as product of d univariate functions whose rth weak derivative is bounded by M. The recovery problem is known to suffer from the curse of dimensionality for $M \geq 2^r r!$. For smaller M, a randomized algorithm is known which breaks the curse. We construct a deterministic algorithm which is even less costly. In fact, we completely characterize the tractability of this problem by distinguishing three different ranges of the parameter M.

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Keywords: High dimensional approximation; Rank one tensors; Worst case error; Tractability; Curse of dimensionality; Dispersion

1. Introduction

Suppose we know that a d-variate function f is the product of d univariate functions with a certain smoothness. How many function values do we need to capture f up to some error $\varepsilon \in (0, 1)$ in the uniform norm? This question has been posed and investigated in the work of

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Bachmayr, Dahmen, DeVore and Grasedyck [2]. The hope is that the structural knowledge about f allows for efficient deterministic approximation schemes in high-dimensional settings. More precisely, it is assumed that f is contained in the class of rank one tensors given by

$$F_{r,M}^d = \Big\{ \bigotimes_{i=1}^d f_i \mid f_i : [0,1] \to [-1,1],\ \|f_i^{(r)}\|_\infty \le M \Big\}$$

for smoothness parameters $r \in \mathbb{N}$ and $M > 0$. Here, $f_i^{(r)}$ denotes the rth weak derivative of f_i. In particular, it is assumed that f_i is contained in the class $W_\infty^r([0,1])$ of univariate functions which have r weak derivatives in $L_\infty([0,1])$.

It is proven in [7] that for $M \ge 2^r r!$ this problem suffers from the curse of dimensionality: To ensure an error smaller than ε, any deterministic algorithm must use exponentially many function values with respect to the dimension. Even for randomized methods, the curse is present. For $M < 2^r r!$ however, a randomized algorithm is constructed which does not require exponentially many function values. We are driven by the question whether the same is possible with a deterministic algorithm. We give an affirmative answer to this question. In fact, we explicitly construct and analyze deterministic algorithms for different ranges of the smoothness parameters. We use the following terminology.

The worst case error of an algorithm A on the class $F_{r,M}^d$ is given by

$$\mathrm{e}(A) := \sup_{f \in F_{r,M}^d} \|f - A(f)\|_\infty.$$

The number of function values used by A for the input f is denoted by $\mathrm{cost}(A, f)$. The worst case cost of A is given by

$$\mathrm{cost}(A) := \sup_{f \in F_{r,M}^d} \mathrm{cost}(A, f).$$

A deterministic algorithm is already constructed in [2]. It achieves the worst case error ε while using at most

$$C_{r,d}\, M^{d/r} \varepsilon^{-1/r}$$

function values of f, see [2, Theorem 5.1]. This number behaves optimally as a function of ε. However, the constant $C_{r,d}$ and hence the number of function values grows super-exponentially with d for any $M > 0$ and $r \in \mathbb{N}$. For the algorithm, the following observation of Bachmayr, Dahmen, DeVore and Grasedyck is crucial. If we know some $z^* \in [0,1]^d$ with $f(z^*) \neq 0$, we can construct a method $I_m(z^*, \cdot)$ that uses m function values and satisfies

$$\big\| I_m(z^*, f) - f \big\|_\infty \le \varepsilon, \tag{1}$$

if we choose

$$m = \big\lfloor C_{r,M}\, d^{1+1/r} \varepsilon^{-1/r} \big\rfloor. \tag{2}$$

Here, $C_{r,M}$ is a positive constant which only depends on r and M. For example, one can choose $C_{r,M} = 4 \max\{1, C_1(r) M\}^{1/r}$ with $C_1(r)$ as in [2, Section 2]. Roughly speaking, the knowledge of a non-zero of f allows us to reduce the problem to d univariate approximation problems which can, for example, be treated by the use of polynomial interpolation. With this observation at hand, the authors of [2] use an approximation scheme of the following type: