Advances in Mathematics 338 (2018) 1–40

Contents lists available at ScienceDirect

Advances in Mathematics

www.elsevier.com/locate/aim

Low rank compact operators and Tingley's problem

Francisco J. Fernández-Polo, Antonio M. Peralta *

Departamento de Análisis Matemático, Facultad de Ciencias, Universidad de Granada, 18071 Granada, Spain

ARTICLE INFO

Article history:
Received 2 December 2016
Received in revised form 31 March 2018
Accepted 21 August 2018
Available online xxxx
Communicated by Dan Voiculescu

MSC:
primary 47B49
secondary 46A22, 46B20, 46B04, 46A16, 46E40

Keywords:
Tingley's problem
Extension of isometries
(weakly compact) JB*-triples
Compact operators
Spin spaces
Cartan factors

ABSTRACT

Let E and B be arbitrary weakly compact JB*-triples whose unit spheres are denoted by $S(E)$ and $S(B)$, respectively. We prove that every surjective isometry $f : S(E) \to S(B)$ admits an extension to a surjective real linear isometry $T : E \to B$. This is a complete solution to Tingley's problem in the setting of weakly compact JB*-triples. Among the consequences, we show that if $K(H, K)$ denotes the space of compact operators between arbitrary complex Hilbert spaces H and K, then every surjective isometry $f : S(K(H, K)) \to S(K(H, K))$ admits an extension to a surjective real linear isometry $T : K(H, K) \to K(H, K)$.

1. Introduction

It does not seem an easy task to write an introductory paragraph for a problem which has been open since 1987. As in most important problems, the precise question is easy to pose and reads as follows: Let X and Y be normed spaces, whose unit spheres are denoted

* Corresponding author.
E-mail addresses: pacopolo@ugr.es (F.J. Fernández-Polo), aperalta@ugr.es (A.M. Peralta).

https://doi.org/10.1016/j.aim.2018.08.018

by $S(X)$ and $S(Y)$, respectively. Suppose $f : S(X) \to S(Y)$ is a surjective isometry. The so-called *Tingley's problem* asks whether f can be extended to a real linear (bijective) isometry $T : X \to Y$ between the corresponding spaces (see [42]).

The problem was named after D. Tingley proved in [42, THEOREM, page 377] that every surjective isometry $f : X \to Y$ between the unit spheres of two finite dimensional spaces satisfies $f(-x) = -f(x)$ for every $x \in S(X)$.

Readers interested in a classic motivation, can sail back to the celebrated Mazur–Ulam theorem asserting that every surjective isometry between two normed spaces over $\mathbb{R}$ is a real affine function. In a subsequent paper, P. Mankiewicz established in [32] that, given two convex bodies $V \subset X$ and $W \subset Y$, every surjective isometry g from V onto W can be uniquely extended to an affine isometry from X onto Y. Consequently, every surjective isometry between the closed unit balls of two Banach spaces X and Y extends uniquely to a real linear isometric isomorphism from X into Y.

With this historical background in mind, Tingley's problem asks whether the conclusion in Mankiewicz's theorem remains true when we deal with the unit sphere whose interior is empty. In other words, when an isometric identification of the unit spheres of two normed spaces can produce an isometric (linear) identification of the spaces.

As long as we know, Tingley's problem remains open even for surjective isometries between the unit spheres of a pair of 2 dimensional Banach spaces. However, positive answers have been established in a wide range of classical Banach spaces. In an interesting series of papers, G.G. Ding proved that Tingley's problem admits a positive answer for every surjective isometry $f : S(\ell^p(\Gamma)) \to S(\ell^p(\Delta))$ with $1 \leq p \leq \infty$ (see [10–12] and [13]). More recently, D. Tan showed that the same conclusion remains true for every surjective isometry $f : S(L^p(\Omega, \Sigma, \mu)) \to S(Y)$, where (Ω, Σ, μ) is a σ-finite measure space, $1 \leq p \leq \infty$, and Y is a Banach space (compare [37,38] and [39]). A result of R.S. Wang in [43] proves that for each pair of locally compact Hausdorff spaces L_1 and L_2, every surjective isometry $f : S(C_0(L_1)) \to S(C_0(L_2))$ admits an extension to a surjective real linear isometry from $C_0(L_1)$ onto $C_0(L_2)$. V. Kadets and M. Martín gave another positive answer to Tingley's problem in the case of finite dimensional polyhedral Banach spaces (see [27]). The surveys [14] and [44] contain a detailed revision of these results and additional references.

In the setting of C*-algebras, R. Tanaka recently establishes in [40] that every surjective isometry from the unit sphere of a finite dimensional C*-algebra N into the unit sphere of another C*-algebra M admits a unique extension to a surjective real linear isometry from N onto M. More recently, Tanaka also proves in [41] that the same conclusion holds when N and M are finite von Neumann algebras.

As a result of a recent collaboration between the second author of this note and R. Tanaka (see [36]), new positive answers to Tingley's problem have been revealed for spaces of compact operators. Concretely, denoting by $K(H)$ the C*-algebra of all compact operators on a complex Hilbert space H, it is shown that for every pair of complex Hilbert spaces H and H', every surjective isometry $f : S(K(H)) \to S(K(H'))$ admits a unique extension to a real linear isometry T from $K(H)$ onto $K(H')$; and the same con-