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Partial differential equations

Self-adjoint and skew-symmetric extensions of the Laplacian with singular Robin boundary condition

Extensions self-adjointes et anti-symétriques du laplacien, avec condition à la frontière de type Robin singulière

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ABSTRACT

We study the Laplacian in a bounded domain, with a varying Robin boundary condition singular at one point. The associated quadratic form is not semi-bounded from below, and the corresponding Laplacian is not self-adjoint, it has a residual spectrum covering the whole complex plane. We describe its self-adjoint extensions and exhibit a physically relevant skew-symmetric one. We approximate the boundary condition, giving rise to a family of self-adjoint operators, and we describe its spectrum by the method of matched asymptotic expansions. A part of the spectrum acquires a strange behavior when the small perturbation parameter $\varepsilon > 0$ tends to zero, namely it becomes almost periodic in the logarithmic scale $|\ln \varepsilon|$, and in this way "wanders" along the real axis at a speed $O(\varepsilon^{-1})$.

RÉSUMÉ

Nous étudions le laplacien dans un domaine borné, avec une condition à la frontière de type Robin, variable et singulière en un point. La forme quadratique associée n'est pas bornée inférieurement, et le laplacien correspondant n'est pas self-adjoint ; son spectre résiduel couvre entièrement le plan complexe. Nous décrivons ses extensions self-adjointes et nous en montrons une anti-symétrique, pertinente en physique. Nous approchons la condition de frontière à l'aide d'une famille d'opérateurs self-adjoints et nous décrivons son spectre par la méthode d'appariement des développements asymptotiques. Une partie du spectre adopte un comportement étrange quand le paramètre $\varepsilon > 0$ de petite perturbation tend vers zéro ; précisément, il devient presque périodique en échelle logarithmique $|\log(\varepsilon)|$, et ainsi « erre » le long de l'axe réel à une vitesse $O(\varepsilon^{-1})$.

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1. Description of the singular problem

In a domain $\Omega \subset \mathbb{R}^2$ enveloped by a smooth simple contour $\partial\Omega$, we consider the Laplacian with a Robin-type boundary condition $a\partial_n u - u = 0$. Here, a is a continuous function defined on $\partial\Omega$, and ∂_n denotes the outward normal derivative to $\partial\Omega$.

If a is positive on $\partial\Omega$, the quadratic form $H^1(\Omega) \ni u \mapsto \|\nabla u; L^2(\Omega)\|^2 - \|a^{-1/2}u, L^2(\partial\Omega)\|^2$ is naturally associated with this problem and, in view of the compact imbedding $H^1(\Omega) \subset L^2(\partial\Omega)$, this form is semi-bounded and closed, and thus defines a self-adjoint operator with compact resolvent. Therefore, the spectrum is an unbounded sequence of real eigenvalues accumulating at $+\infty$. Note that the first eigenvalue is negative, and goes to $-\infty$ if a is a small positive constant, see [5].

Let a become zero at a point $x_0 \in \partial\Omega$. In this note, we will mainly consider the case where a vanishes at order one, i.e. admits the Taylor formula

$$a(s) = a_0 s + O(s^2), \quad s \to 0 \quad \text{with } a_0 > 0, \tag{1}$$

where s is a curvilinear abscissa starting at x_0. For convenience, we denote $b_0 := a_0^{-1}$.

Since we assume a to be continuous, there should exist at least one other point where a vanishes. However, several points of vanishing do not bring any new effect, and we replace the problem by another one: we assume that $\partial\Omega$ is the union of two smooth curves Γ_1 and Γ_2 which meet perpendicularly, with x_0 in the interior of Γ_1, and that a vanishes only at x_0, according to (1). We complete the Robin boundary condition on Γ_1 by a Neumann boundary condition on Γ_2. Therefore, our spectral problem is

$$\begin{cases} -\Delta u = \lambda u \ \text{ on } \ \Omega, \\ a\partial_n u - u = 0 \ \text{ on } \ \Gamma_1, \ \text{ and } \ \partial_n u = 0 \ \text{ on } \ \Gamma_2. \end{cases} \tag{2}$$

The associated quadratic form is defined on $D(q) := \{u \in H^1(\Omega), a^{-\frac{1}{2}} u_{|\Gamma_1} \in L^2(\Gamma_1)\}$ as follows

$$D(q) \ni u \mapsto \int_\Omega |\nabla u|^2 \mathrm{d}x - \int_{\Gamma_1} a^{-1}|u|^2 \mathrm{d}s.$$

It is not semi-bounded anymore. Thus, there is no canonical way for defining a self-adjoint operator associated with problem (2). The natural definition becomes the operator A_0 acting as $-\Delta$ on the domain

$$D(A_0) := \{u \in D(q), \Delta u \in L^2(\Omega), a\partial_n u - u = 0 \text{ on } \Gamma_1, \partial_n u = 0 \text{ on } \Gamma_2\}. \tag{3}$$

Such a problem was studied in [1,6] in a model half-disk, for which the eigenvalue equation had the advantage to decouple in polar coordinates. The authors found that A_0 is non-self-adjoint. In [6], they clarified the "paradox" from [1] stating that, for any $\lambda \in \mathbb{C}$, problem (2) has a nontrivial solution, by showing that the spectrum of A_0^* is residual and coincides with the complex plane.

A three-dimensional version of this spectral problem appears also in the modeling of a spinless particle moving in two thin films with a one-contact point, and has been studied in a model domain in [3].

2. Goal and results

In this note, we explain how to find extensions of A_0 and give a better understanding of their spectrum, arguing with an asymptotic approach. We also exhibit a relevant skew-symmetric extension using a physical argument.

The domain of A_0^* is

$$D(A_0^*) := \{u \in L^2(\Omega), \Delta u \in L^2(\Omega), a\partial_n u - u = 0 \text{ on } \Gamma_1, \ \partial_n u = 0 \text{ on } \Gamma_2\}.$$

To understand how different $D(A_0^*)$ is from $D(A_0)$, we exhibit two possible singular behaviors for functions in $D(A_0^*)$ at the point x_0. Using Kondratiev's theory [4], we investigate a model problem in a half-plane and, as a result, describe $D(A_0^*)$. We deduce, going over the domain Ω, that the deficiency indices of A_0 are (1,1), and we classify its self-adjoint extensions using a parametrization $\theta \mapsto \mathrm{e}^{\mathrm{i}\theta}$ of the unit circle $\mathbb{S}^1 \subset \mathbb{C}$. The description of A_0^* allows us also to introduce a natural skew-symmetric extension of A_0 corresponding to a Sommerfeld radiation condition at x_0.

Next, we approach our problem by a family of self-adjoint operators by choosing a suitable perturbation of the Robin coefficient a. This is done by means of the non-vanishing discontinuous function

$$a_\varepsilon(s) = a_0 \operatorname{sign}(s)\varepsilon + a(s) \tag{4}$$

satisfying $\inf_{\Gamma_1} |a_\varepsilon| = \varepsilon$, and we study the discrete spectrum of the associated Robin Laplacian as $\varepsilon \to 0$. Using the method of matched asymptotic expansions, we find that its spectrum is related to the eigenvalues of self-adjoint extensions, with a parameter θ_ε oscillating in the logarithmic scale as $\varepsilon \to 0$.