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Partial differential equations

Solutions to a nonlinear Neumann problem in three-dimensional exterior domains

Solutions d'un problème de Neumann non linéaire dans des domaines extérieurs en dimension 3

Adélaïde Olivier [a], Olivier Rey [b]

[a] *Laboratoire de mathématiques d'Orsay, Université Paris-Sud, CNRS, Université Paris-Saclay, 91405 Orsay, France*
[b] *IHPST, Université Paris-1 Panthéon-Sorbonne, CNRS, 13, rue du Four, 75006 Paris, France*

A R T I C L E I N F O

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Presented by Jean-Michel Coron

A B S T R A C T

We prove the existence of multipeak solutions to a nonlinear elliptic Neumann problem involving nearly critical Sobolev exponent, in three-dimensional exterior domains.

R É S U M É

Nous démontrons, pour un problème elliptique de Neunmann avec non-linéarité presque critique, dans un domaine extérieur en dimension trois, l'existence de solutions qui se concentrent en plusieurs points de la frontière lorsque la non-linéarité devient critique.

To Prof. Norman Dancer

1. Introduction and results

There have been innumerable articles devoted, over the last three decades, to the study of elliptic partial differential equations of second order with critical nonlinearity. One thing, however, is to be noticed: virtually all articles consider problems in bounded domains. A work like Yan's one [15] is an exception. In that paper, Yan considers the following Neumann problem:

$$\begin{cases} -\Delta u = u^{2^*-1-\varepsilon} \quad , u > 0 & \text{in } \mathbb{R}^N \setminus \Omega \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

E-mail addresses: adelaide.olivier@math.u-psud.fr (A. Olivier), olivier.rey@univ-paris1.fr (O. Rey).

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where Ω is a smooth and bounded domain in $\mathbb{R}^N$, $N \geq 3$, such that $R^N \setminus \Omega$ is connected, $2^* = 2N/(N-2)$ is the limiting Sobolev exponent for the embedding of the Sobolev space $W^{1,2}(\Omega)$ into the $L^p(\Omega)$-spaces, and ε is a strictly positive number, assumed to be small. Although formally, the problem is subcritical, it is asymptotically critical, and the techniques to be used to study it as ε goes to zero are the same as those to be used in the case of critical nonlinearities.

Considering exterior domains is all but arbitrary. Indeed, various models lead to study the equation $-\Delta u = u^p$ in domains with small holes. As the size of these holes goes to zero, the limiting problem which is obtained through a rescaling centered on one of the holes looks precisely as (1.1).

In order to state the results proved by Yan, as well as those that we propose to establish, some notations have first to be introduced. For $\lambda \in \mathbb{R}^*_+$ and $x \in \mathbb{R}^N$, we denote by $U_{\lambda,x}$ the function defined in $\mathbb{R}^N$ by

$$U_{\lambda,x}(y) = [N(N-2)]^{\frac{N-2}{4}} \frac{\lambda^{\frac{N-2}{2}}}{\left(1+\lambda^2|y-x|^2\right)^{\frac{N-2}{2}}}. \tag{1.2}$$

The $U_{\lambda,x}$'s are the only nontrivial solutions to the equation $-\Delta U = U^{2^*-1}$, $U \geq 0$ in $\mathbb{R}^N$ (see for example [2], [13] or [7]) and induce, as λ goes to infinity, a lack of compactness of the embedding of $W^{1,2}$ into L^{2^*}. In the following, $D^{1,2}(\mathbb{R}^N \setminus \Omega)$ refers to the completion of the set of smooth functions with compact support in $\mathbb{R}^N \setminus \Omega$ for the norm

$$\|u\| = < u, u >^{1/2}$$

with

$$< u, v > = \int_{R^N \setminus \Omega} \nabla u \cdot \nabla v.$$

Lastly, we denote by $H(y)$ the mean curvature of $\partial\Omega$ at a point y of this boundary. Yan proves:

Theorem 1.1. [15]. *Assume that* $N \geq 4$.

(1) [Case of a positive local maximum of H.] *Suppose that S is a connected subset of $\partial\Omega$ satisfying: $H(y) = H_m > 0$ for any $y \in S$; and there exists $\delta > 0$ such that $H(y) < H_m$ for any $y \in S_\delta \setminus S$, and H has no critical point in $S_\delta \setminus S$, with $S_\delta = \{y \in \partial\Omega$ s.t. $d(x, y) \leq \delta\}$. Then, for any positive integer k, there exists $\varepsilon_0 > 0$ such that for any $\varepsilon_0 \in (0, \varepsilon_0)$, (1.1) has a solution*

$$u_\varepsilon = \sum_{i=1}^{k} \alpha_{\varepsilon,i} U_{\lambda_{\varepsilon,i}, x_{\varepsilon,i}} + v_\varepsilon \tag{1.3}$$

where, as ε goes to zero

$$\alpha_{\varepsilon,i} \to 1$$
$$\varepsilon\lambda_{\varepsilon,i} \to c^* H_m \quad c^* \textit{ a positive constant depending on N only}$$
$$x_{\varepsilon,i} \in S_\delta \quad \text{and} \quad x_{\varepsilon,i} \to x_i \in S$$

for any i, $i \leq i \leq k$, and

$$v_\varepsilon \to 0 \quad \text{in} \quad D^{1,2}(\mathbb{R}^N \setminus \Omega).$$

(2) [Case of a positive local minimum of H.] *Suppose that S is a connected subset of $\partial\Omega$ satisfying: $H(y) = H_m > 0$ for any $y \in S$; and there exists $\delta > 0$ such that $H(y) > H_m$ for any $y \in S_\delta \setminus S$, and H has no critical point in $S_\delta \setminus S$, with $S_\delta = \{y \in \partial\Omega \quad s.t. \quad d(x, y) \leq \delta\}$. Then, there exists $\varepsilon_0 > 0$ such that, for any $\varepsilon_0 \in (0, \varepsilon_0)$, (1.1) has a solution*

$$u_\varepsilon = \alpha_\varepsilon U_{\lambda_\varepsilon, x_\varepsilon} + v_\varepsilon \tag{1.4}$$

where $\alpha_{\varepsilon,i} \to 1$, $\varepsilon\lambda_\varepsilon \to c^* H_m$, $x_\varepsilon \to x_0 \in S$ and $v_\varepsilon \to 0$ in $D^{1,2}(\mathbb{R}^N \setminus \Omega)$ as ε goes to zero.

The arguments developed by Yan to prove Theorem 1.1 allow him to consider also the problem

$$\begin{cases} -\Delta u + \mu u = u^{2^*-1} \quad , u > 0 & \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \end{cases} \tag{1.5}$$

where μ is a positive number assumed to be large. Yan proves, for $N \geq 5$ and a positive local minimum of H, an equivalent of Theorem 1.1 (1) as μ goes to infinity ($1/\mu$ plays a role similar to that of ε previously). Such a result has been extended to the cases $N = 3, 4$ by Wei and Yan in [14]. On the other hand, it has never been demonstrated until now that the statement

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