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Functional analysis/Dynamical systems

On sofic groupoids and their full groups

Sur les groupoïdes sofiques et leurs groupes pleins

Luiz Cordeiro [1]

University of Ottawa, Canada

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ABSTRACT

We prove that the class of sofic groupoids is stable under several measure-theoretic constructions. In particular, we show that virtually sofic groupoids are sofic. We answer a question of Conley, Kechris, and Tucker-Drob by proving that an aperiodic pmp groupoid is sofic if and only if its full group is metrically sofic.

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RÉSUMÉ

Nous démontrons dans cette note que plusieurs constructions de théorie de la mesure préservent la classe des groupoïdes sofiques. En particulier, nous montrons qu'un sous-groupoïde virtuellement sofique est sofique. Nous répondons aussi à une question de Conley, Kechris et Tucker-Drob en démontrant que, pour qu'un groupoïde apériodique muni d'une mesure de probabilité invariante soit sofique, il est nécessaire et suffisant que son groupe plein soit métriquement sofique.

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1. Introduction

The notion of soficity for groups was introduced by Gromov [11] in his work on symbolic dynamics. In 2010, Elek and Lippner [7] introduced soficity for equivalence relations in the same spirit as Gromov's original definition, i.e. an equivalence relation R, induced by some action of the free group $\mathbb{F}_\infty$, is sofic if the Schreier graph of the $\mathbb{F}_\infty$-space X can be approximated, in a suitable sense, by Schreier graphs of finite $\mathbb{F}_\infty$-spaces.

Alternative definitions by Ozawa [16] and Păunescu [17] describe soficity at the level of the so-called full semigroup of R, which can be immediately generalized to groupoids. We will describe general elementary techniques to deal with (abstract) sofic groupoids.

E-mail address: lcord081@uottawa.ca.

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1.1. Probability measure-preserving groupoids and full semigroups

We will follow the notations of [5]: given a groupoid G, the source and range maps will be, respectively, $s(g)=g^{-1}g$ and $r(g)=gg^{-1}$ for $g\in G$, and the unit space of G will be denoted $G^{(0)}$.

A *discrete measurable groupoid* is a groupoid G endowed with a standard Borel space structure such that the product and inversion maps are Borel, and such that $s^{-1}(x)$ is countable for every $x\in G^{(0)}$.

The *Borel full semigroup* of a discrete measurable groupoid G is the set $[[G]]_B$ of Borel subsets $\alpha\subseteq G$ such that the restrictions $s|_\alpha$ and $r|_\alpha$ of the source and range maps are injections, and thus Borel isomorphisms onto their respective images ([13, Theorem 15.2]).

$[[G]]_B$ is an inverse monoid with the usual product and inverse of sets, namely

$$\alpha\beta=\left\{ab:(a,b)\in(\alpha\times\beta)\cap G^{(2)}\right\},\qquad \alpha^{-1}=\left\{a^{-1}:a\in\alpha\right\}$$

and $G^{(0)}$ is the unit of $[[G]]_B$, which we will instead denote by $G^{(0)}=1$ when no confusion arises.

A *probability measure-preserving* (*pmp*) groupoid is a discrete measurable groupoid G with a Borel probability measure μ on $G^{(0)}$ satisfying $\mu(s(\alpha))=\mu(r(\alpha))$ for all $\alpha\in[[G]]_B$. We write (G,μ) for a pmp groupoid when we need the measure μ to be explicit. The measure μ induces a pseudometric d_μ on $[[G]]_B$ via

$$d_\mu(\alpha,\beta)=\mu(s(\alpha\triangle\beta))=\mu(r(\alpha\triangle\beta)).$$

The *trace* of $\alpha\in[[G]]_B$ is defined as $\operatorname{tr}(\alpha)=\mu(\alpha\cap G^{(0)})$. In fact, the trace and the pseudometric above, along with the semigroup operation, determine each other: for example, the unit 1 of $[[G]]_B$ is the only element of trace 1, and

$$d_\mu(\alpha,\beta)=\operatorname{tr}(\alpha^{-1}\alpha)+\operatorname{tr}(\beta^{-1}\beta)-\operatorname{tr}(\alpha^{-1}\alpha\beta^{-1}\beta)-\operatorname{tr}(\beta^{-1}\alpha)$$

and similarly one can write the trace in terms of the pseudometric d_μ.

The (measured) *full semigroup* of a pmp groupoid (G,μ) is the quotient metric space $[[G]]$ (or $[[G]]_\mu$ to make μ explicit) of $[[G]]_B$ under the pseudometric d_μ. In fact, $[[G]]$ is an inverse semigroup, with the quotient operation endowed from $[[G]]_B$, and the trace map $\operatorname{tr}:[[G]]_B\to\mathbb{R}$ factors through a map on $[[G]]$. We will not distinguish $[[G]]_B$ and $[[G]]$ unless strictly necessary.

The *Borel full group* $[G]_B$ of a discrete measurable groupoid G is the set of those $\alpha\in[[G]]_B$ with $s(\alpha)=r(\alpha)=G^{(0)}$, and, when G is pmp, the image of $[G]_B$ in $[[G]]$, denoted $[G]$ or $[G]_\mu$, is called the (measured) *full group* of G.

Definition 1.1. A subset A of a pmp groupoid (G,μ) is called *null* if $\mu(s(A))=0$ (equivalently, $\mu(r(A))=0$), and *conull* if its complement $G\setminus A$ is null. A property of the points of G is said to hold a.e. (almost everywhere) if it holds on a conull subset.

Example 1.2. Let R be a countable Borel equivalence relation on a standard probability space (X,μ), and suppose that μ is invariant (see [8]). We can see R as a pmp groupoid as follows: the product is defined by $(x,y)(y,z)=(x,z)$. The unit space of R is the diagonal $\{(x,x):x\in X\}$, which we identify with X and endow with the probability measure μ. The Borel full semigroup of R can be identified with the semigroup of partial Borel isomorphisms $f:A\to B$, $A,B\subseteq X$, for which $(f(x),x)\in R$ for all $x\in A$, by associating such f with the inverse of its graph, $\{(f(x),x):x\in X\}$. The pmp groupoids that are isomorphic (in the measure-theoretic sense) to one constructed in this way are called *principal groupoids*.

Example 1.3. Let Y be a finite set and Y^2 the largest equivalence relation on Y, endowed with the usual (discrete) Borel structure. The only probability measure on Y that makes Y^2 pmp is the normalized counting measure: $\mu_\#(A)=|A|/|Y|$. We denote the associated metric by $d_\#$ and call it the *normalized Hamming distance*.

Note that if Y and Z are finite sets, then the map $[[Y^2]]\ni\alpha\mapsto\alpha\times(Z^2)^{(0)}\in[[Y^2\times Z^2]]$ is a trace-preserving embedding. The map $(y_1,y_2,z_1,z_2)\mapsto(y_1,z_1,y_2,z_2)$ is a measure-preserving isomorphism between the groupoids $Y^2\times Z^2$ and $(Y\times Z)^2$, which induces a trace-preserving isomorphism between the respective two full semigroups. Therefore, if Y and Z are finite sets, there are a finite set W and trace-preserving embeddings from $[[Y^2]]$ and $[[Z^2]]$ into $[[W^2]]$.

Definition 1.4. A *sofic approximation* of a pmp groupoid G is a sequence of maps $\pi=\left\{\pi_k:[[G]]\to[[Y_k^2]]\right\}$, where Y_k are finite sets, such that for all $\alpha,\beta\in[[G]]$,

(i) $\lim_{k\to\infty}\operatorname{tr}(\pi_k(\alpha))=\operatorname{tr}(\alpha)$;
(ii) $\lim_{k\to\infty}d_\#(\pi_k(\alpha\beta),\pi_k(\alpha)\pi_k(\beta))=0$.

A pmp groupoid G is *sofic* if it admits a sofic approximation.

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