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Algebraic geometry/Topology

Milnor and Tjurina numbers for a hypersurface germ with isolated singularity

Nombres de Milnor et Tjurina pour les germes d'hypersurfaces à singularité isolée

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A R T I C L E I N F O

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A B S T R A C T

Assume that $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ is an analytic function germ at the origin with only isolated singularity. Let μ and τ be the corresponding Milnor and Tjurina numbers. We show that $\dfrac{\mu}{\tau} \leq n$. As an application, we give a lower bound for the Tjurina number in terms of n and the multiplicity of f at the origin.

R É S U M É

Soit $f : (\mathbf{C}^n, 0) \to (\mathbf{C}, 0)$ un germe de fonction analytique au voisinage de l'origine avec une seule singularité isolée. Soient μ et τ les nombres de Milnor et Tjurina correspondants. Nous montrons que $\frac{\mu}{\tau} \leq n$. Comme application, nous donnons une minoration du nombre de Tjurina en fonction de n et de la multiplicité de f à l'origine.

1. Main result

Assume that $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ is an analytic function germ at the origin with only isolated singularity. Set $X = f^{-1}(0)$. Let $S = \mathbb{C}\{x_1, \ldots, x_n\}$ denote the formal power series ring. Set $J_f = (\partial f/\partial x_1, \ldots, \partial f/\partial x_n)$ as the Jacobian ideal. Then the Milnor and Tjurina algebras are defined as

$$M_f = S/J_f, \text{ and } T_f = S/(J_f, f).$$

Since X has isolated singularities, M_f and T_f are finite dimensional $\mathbb{C}$-vector spaces. The corresponding dimension μ and τ are called the Milnor and Tjurina numbers, respectively. It is clear that $\mu \geq \tau$.

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Consider the following long exact sequence of $\mathbb{C}$-algebras:

$$0 \to \mathrm{Ker}(f) \to M_f \xrightarrow{f} M_f \to T_f \to 0 \tag{1}$$

where the middle map is multiplication by f, and $\mathrm{Ker}(f)$ is the kernel of this map. Then $\dim_{\mathbb{C}} \mathrm{Ker}(f) = \tau$.

Recall a well-known result given by J. Briançon and H. Skoda in [1],

$$f^n \in J_f,$$

which shows that $f^n = 0$ in M_f, i.e. $(f^{n-1}) \subset \mathrm{Ker}(f)$. Here (f^{n-1}) is the ideal in M_f generated by f^{n-1}. The following theorem is a direct application of this result.

Theorem 1.1. *Assume that* $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ *is an analytic function germ at the origin with only isolated singularity. Then,*

$$\frac{\mu}{\tau} \le n.$$

Moreover, $\frac{\mu}{\tau} = n$, *if and only if,* $\mathrm{Ker}(f) = (f^{n-1})$.

Proof. Since $f^n = 0$ in M_f, we have the following finite decreasing filtration:

$$M_f \supset (f) \supset (f^2) \supset \cdots \supset (f^{n-1}) \supset (f^n) = 0$$

where (f^i) is the ideal in M_f generated by f^i.

Consider the following long exact sequence:

$$0 \to \mathrm{Ker}(f) \cap (f^i) \to (f^i) \xrightarrow{f} (f^i) \to (f^i)/(f^{i+1}) \to 0 \tag{2}$$

where the middle map is multiplication by f. Then,

$$\dim_{\mathbb{C}}\{(f^i)/(f^{i+1})\} = \dim_{\mathbb{C}}\{\mathrm{Ker}(f) \cap (f^i)\} \le \dim_{\mathbb{C}} \mathrm{Ker}(f) = \tau.$$

Therefore,

$$\mu = \dim_{\mathbb{C}} M_f = \dim_{\mathbb{C}} T_f + \sum_{i=1}^{n-1} \dim_{\mathbb{C}}\{(f^i)/(f^{i+1})\} \le n \cdot \tau.$$

$\frac{\mu}{\tau} = n$ if and only if, for any $1 \le i \le n-1$, $\mathrm{Ker}(f) \cap (f^i) = \mathrm{Ker}(f)$, i.e. $\mathrm{Ker}(f) \subset (f^i)$. On the other hand, $(f^{n-1}) \subset \mathrm{Ker}(f)$. Hence, $\mathrm{Ker}(f) = (f^{n-1})$. □

K. Saito showed ([8]) that $\frac{\mu}{\tau} = 1$ holds, if and only if, f is weighted homogeneous, i.e. analytically equivalent to such a polynomial. It leads to the following natural question.

Question 1.2. Is this upper bound of $\frac{\mu}{\tau}$ optimal? When can the optimal upper bound be obtained?

Remark 1.3. Recently, A. Dimca and G.-M. Greuel showed ([3, Theorem 1.1]) that the upper bound $\frac{\mu}{\tau} \le 2$ can never be achieved for the isolated plane curve singularity case unless f is smooth at the origin. Moreover, they gave ([3, Example 4.1]) a sequence of isolated plane curve singularity with the ratio $\frac{\mu}{\tau}$ strictly increasing towards 4/3. In particular, the singularities can be chosen to be all either irreducible, or consisting of smooth branches with distinct tangents. Based on these computations, they asked ([3, Question 4.2]) whether

$$\frac{\mu}{\tau} < 4/3$$

for any isolated plane curve singularity.

Example 1.4. It is clear that $\frac{\mu}{\tau} > n-1$ implies that $f^{n-1} \notin J_f$.

Consider the function germ:

$$f = (x_1 \cdots x_n)^2 + x_1^{2n+2} + \cdots + x_n^{2n+2},$$