C. R. Acad. Sci. Paris, Ser. I ••• (••••) •••–•••

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

www.sciencedirect.com

Probability theory

A note on the quasi-ergodic distribution of one-dimensional diffusions

Une note sur la distribution quasi ergodique des diffusions en dimension 1

Guoman He

School of Mathematics and Statistics, Hunan University of Commerce, Changsha, Hunan 410205, PR China

A R T I C L E I N F O

Article history:
Received 15 March 2018
Accepted after revision 18 July 2018
Available online xxxx

Presented by Jean-François Le Gall

A B S T R A C T

In this note, we study quasi-ergodicity for one-dimensional diffusions on $(0,\infty)$, where 0 is an exit boundary and $+\infty$ is an entrance boundary. Our main aim is to improve some results obtained by He and Zhang (2016) [3]. In simple terms, the same main results of the above paper are obtained with more relaxed conditions.

R É S U M É

Nous étudions la quasi-ergodicité des diffusions unidimensionnelles sur $]0,\infty[$, où 0 est une frontière de sortie et ∞ une frontière d'entrée. Notre but est d'améliorer des résultats obtenus par He and Zhang (2016) [3]. Ainsi, nous retrouvons les résultats principaux de ce texte sous des hypothèses moins restrictives.

1. Introduction

Let $X=(X_t, t\geq 0)$ be a one-dimensional drifted Brownian motion on $(0,\infty)$, i.e.

$$dX_t = dB_t - \alpha(X_t)\,dt, \qquad X_0 = x > 0, \tag{1.1}$$

where $(B_t, t\geq 0)$ is a standard one-dimensional Brownian motion and $\alpha \in C^1(0,\infty)$. In this paper, α can explode at the origin. There exists a pathwise unique solution to the stochastic differential equation (1.1) up to the explosion time τ.

Associated with α, we consider the following two functions

$$\Lambda(x) = \int_1^x e^{Q(y)}\,dy \quad \text{and} \quad \kappa(x) = \int_1^x e^{Q(y)} \left(\int_1^y e^{-Q(z)} dz \right) dy, \tag{1.2}$$

where $Q(y) := 2\int_1^y \alpha(x)\,dx$. Note that Λ is the scale function of the process X.

E-mail address: hgm0164@163.com.

https://doi.org/10.1016/j.crma.2018.07.009

Please cite this article in press as: G. He, A note on the quasi-ergodic distribution of one-dimensional diffusions, C. R. Acad. Sci. Paris, Ser. I (2018), https://doi.org/10.1016/j.crma.2018.07.009

The other important role will be played by the following measure μ, which is not necessarily finite in this paper, defined on $(0,\infty)$:

$$\mu(\mathrm{d}y) := \mathrm{e}^{-Q(y)}\mathrm{d}y. \tag{1.3}$$

Note that μ is the speed measure of the process X.

Let $T_a := \inf\{0 \le t < \tau : X_t = a\}$ be the hitting time of $a \in (0,\infty)$ for the process X. We denote by $T_\infty = \lim_{n\to\infty} T_n$ and $T_0 = \lim_{n\to\infty} T_{1/n}$. Since α is regular in $(0,\infty)$, then $\tau = \min\{T_0, T_\infty\}$. Let $\mathbb{P}_x$ be the probability measure under which the process X starts at x. For any distribution π on $(0,\infty)$, we will use the notation

$$\mathbb{P}_\pi(\cdot) := \int_0^\infty \mathbb{P}_x(\cdot)\pi(\mathrm{d}x).$$

We shall denote by $\mathbb{E}_x$ (resp. $\mathbb{E}_\pi$) the expectation corresponding to $\mathbb{P}_x$ (resp. $\mathbb{P}_\pi$). We denote by $\mathcal{B}(0,\infty)$ the Borel σ-algebra on $(0,\infty)$, $\mathscr{P}(0,\infty)$ the set of all probability measures on $(0,\infty)$ and $\mathbf{1}_A$ the indicator function of A. We define the inner product

$$\langle f, g\rangle_\mu = \int_0^\infty f(u)g(u)\mu(\mathrm{d}u).$$

In this paper, we will use the following hypothesis (H).

Definition 1.1. We say that hypothesis (H) holds if the following explicit conditions on α, all together, are satisfied:

(H1) for all $x > 0$, $\mathbb{P}_x(\tau = T_0 < T_\infty) = 1$;
(H2) for any $\varepsilon > 0$, $\mu(0,\varepsilon) = \infty$;
(H3) $S = \int_1^\infty \mathrm{e}^{Q(y)} \left(\int_y^\infty \mathrm{e}^{-Q(z)}\mathrm{d}z\right) \mathrm{d}y < \infty$.

It is well known (see, e.g., [4], Chapter VI, Theorem 3.2) that (H1) is equivalent to $\Lambda(\infty) = \infty$ and $\kappa(0^+) < \infty$. According to Feller's classification (see [5, Chapter 15]), if (H1) and (H2) are satisfied, then 0 is an exit boundary; if (H1) and (H3) are satisfied, then $+\infty$ is an entrance boundary.

One of the fundamental problems for a killed Markov process conditioned on survival is to study its long-term asymptotic behavior. In order to understand the behavior of the process before extinction, a relevant object to look at is a so-called quasi-ergodic distribution (see [1]). In this paper, we will study the existence and uniqueness of quasi-ergodic distributions for the one-dimensional diffusion process X satisfying hypothesis (H).

Recently, under the conditions that hypothesis (H) holds and the killed semigroup satisfies intrinsic ultracontractivity, He and Zhang [3] proved that there exists a unique quasi-ergodic distribution for the one-dimensional diffusion process X. The main aim of this note is to show that this conclusion still holds only under hypothesis (H) without the intrinsic ultracontractivity. Our main result is Theorem 3.1 (see Section 3).

2. Preliminaries

Before going to our main result, we give some preliminaries. We denote by $L := \frac{1}{2}\partial_{xx} - \alpha\partial_x$ the infinitesimal generator of the one-dimensional diffusion process X. From [6], we know that L is the generator of a strongly continuous symmetric semigroup of contractions on $\mathbb{L}^2(\mu)$ denoted by $(P_t)_{t\ge 0}$. This semigroup is sub-Markovian, that is, $0 \le P_t f \le 1$ μ-a.e. if $0 \le f \le 1$. Also from [6], we get that when (H1) holds, the semigroup of the process X killed at 0 can be given by

$$P_t f(x) = \mathbb{E}_x[f(X_t), T_0 > t].$$

In this paper, we study quasi-ergodicity for one-dimensional diffusions on $(0,\infty)$, where 0 is an exit boundary and $+\infty$ is an entrance boundary. More formally, the main object of interest of this work would be captured by the following definition.

Definition 2.1. We say that $m \in \mathscr{P}(0,\infty)$ is a quasi-ergodic distribution if there exists a $\pi \in \mathscr{P}(0,\infty)$ such that, for any $A \in \mathcal{B}(0,\infty)$,

$$\lim_{t\to\infty} \mathbb{E}_\pi\left(\frac{1}{t}\int_0^t \mathbf{1}_A(X_s)\,\mathrm{d}s \middle| T_0 > t\right) = m(A).$$

Please cite this article in press as: G. He, A note on the quasi-ergodic distribution of one-dimensional diffusions, C. R. Acad. Sci. Paris, Ser. I (2018), https://doi.org/10.1016/j.crma.2018.07.009