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Degree 1 elements of the Selberg class

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Abstract

We provide a short and simple proof of a beautiful result of Kaczorowski and Perelli classifying the elements of degree one in the Selberg class.

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In [5], Selberg axiomatized properties expected of L-functions and introduced the "Selberg class." We recall that an element F of the Selberg class $\mathscr{S}$ satisfies the following axioms.

Axiom 1. In the half-plane $\sigma > 1$ the function $F(s)$ is given by an absolutely convergent Dirichlet series $\sum_{n=1}^{\infty} a(n)n^{-s}$ with $a(1) = 1$ and $a(n) \ll n^{\varepsilon}$ for every $\varepsilon > 0$.

Axiom 2. There is a natural number m such that $(s-1)^m F(s)$ extends to an analytic function of finite order in the entire complex plane.

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Axiom 3. There is a function $\Phi(s) = Q^s G(s) F(s)$ where $Q > 0$ and

$$G(s) = \prod_{j=1}^{r} \Gamma(\lambda_j s + \mu_j) \quad \text{with} \quad \lambda_j > 0 \text{ and } \operatorname{Re} \mu_j \geqslant 0$$

such that

$$\Phi(s) = \omega \overline{\Phi}(1 - s),$$

where $|\omega| = 1$ and for any function f we denote $\overline{f}(s) = \overline{f(\overline{s})}$. We let $d := 2 \sum_{j=1}^{r} \lambda_j$ denote the "degree" of F.

Axiom 4. We may express $\log F(s)$ by a Dirichlet series

$$\log F(s) = \sum_{n=2}^{\infty} \frac{b(n)}{n^s} \frac{\Lambda(n)}{\log n},$$

where $b(n) \ll n^{\vartheta}$ for some $\vartheta < \frac{1}{2}$. Set $b(n) = 0$ if n is not a prime power.

A fundamental conjecture asserts that the degree of an element in the Selberg class is an integer. From the work of Richert [4] it follows that there are no elements in the Selberg class with degree $0 < d < 1$. This was rediscovered by Conrey and Ghosh [1] who also proved that the Selberg class of degree 0 contains only the constant function 1. Recently, Kaczorowski and Perelli [2] determined the structure of the Selberg class for degree 1 and showed that this consists of the Riemann zeta function and shifts of Dirichlet L-functions. Subsequently, in [3] they showed that there are no elements of the Selberg class with degree $1 < d < \frac{5}{3}$. In this note, we shall give a short and simple proof of Kaczorowski and Perelli's beautiful result on the Selberg class for degree 1.

Theorem. *Suppose F satisfies Axioms* 1–3 *and that the degree of F is* 1. *Then there exists a positive integer q and a real number A such that $a(n)n^{-\mathrm{i}A}$ is periodic* $(\operatorname{mod} q)$. *If in addition F satisfies Axiom* 4 *then there is a primitive Dirichlet character χ'* $(\operatorname{mod} q')$ *such that* $F(s) = L(s + \mathrm{i}A, \chi')$.

We remark that Kaczorowski and Perelli obtain their results without assuming the hypothesis $a(n) \ll n^{\varepsilon}$. We could restructure our proof to avoid this assumption, but have preferred not to do so in the interest of keeping the exposition transparent. Our method may also be modified and combined with the ideas in [3] to give a simplification of their result for $1 < d < \frac{5}{3}$.

Suppose F satisfies Axioms 1–3 and has degree 1. By Stirling's formula we see that for $t \geqslant 1$

$$\frac{\overline{G}(1/2 - \mathrm{i}t)}{G(1/2 + \mathrm{i}t)} = \mathrm{e}^{-\mathrm{i}t \log \frac{t}{2e} + \mathrm{i}\frac{\pi}{4} + \mathrm{i}B} t^{\mathrm{i}A} C^{-\mathrm{i}t} \left(1 + O\left(\frac{1}{t}\right)\right), \tag{1}$$