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A "Generalized Trace Formula" for Bell numbers☆

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Abstract

The nth Bell number B_n is the number of ways to partition a set of n elements into nonempty subsets. We generalize the "trace formula" of Barsky and Benzaghou [1], which asserts that for an odd prime p and an appropriate constant τ_p, the relation $B_n = -\mathrm{Tr}(\vartheta^{n-1-\tau_p})B_{\tau_p}$ holds in $\mathbb{F}_p$, where ϑ is a root of $\widetilde{g}(x) = x^p - x - 1$ and $\mathrm{Tr} : \mathbb{F}_p[\vartheta] \longrightarrow \mathbb{F}_p$ is the trace form. We deduce some new interesting congruences for the Bell numbers, generalizing miscellaneous well-known results including those of Radoux [4].

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1. Generating function

Let $\mathscr{A}$ be any integral domain containing $\mathbb{Z}$. As in [2], we consider the linear operator $\Phi : \mathscr{A}[x] \longrightarrow \mathscr{A}[x]$ which maps the Pochhammer polynomial $(x)_n = x(x-1)\cdots(x-(n-1))$ on x^n and we define the Bell polynomials $B_n(x) = \Phi(x^n)$. With the Stirling numbers of the second kind, we have

$$B_n(x) = \left\{ {n \atop 0} \right\} + \left\{ {n \atop 1} \right\} x^1 + \left\{ {n \atop 2} \right\} x^2 + \cdots + \left\{ {n \atop n} \right\} x^n$$

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and the Bell numbers are simply $B_n = B_n(1) = [\Phi(x^n)]_{x=1}$. The ordinary generating function for Bell polynomials is given by

$$F(x,z) := \sum_{n\geqslant 0} B_n(x) z^n = 1 + \Phi\left(xz \sum_{n\geqslant 0} (xz)^n\right).$$

The relation $x\Phi((x)_n) = x^{n+1} = \Phi((x)_{n+1}) = \Phi(x(x-1)_n)$ holds for every integer n and by linearity, we deduce that $\Phi(xf(x)) = x\Phi(f(x+1))$ for any polynomial $f(x) \in \mathscr{A}[x]$.

Using this fact, we get

$$F(x,z) = 1 + xz\ \Phi\left(\sum_{n\geqslant 0} ((x+1)z)^n\right) = 1 + xz\ \Phi\left(\frac{1}{1-z-xz}\right)$$

and therefore

$$F(x,z) = 1 + \frac{xz}{1-z}\ \Phi\left(\frac{1}{1-xz/(1-z)}\right) = 1 + \frac{xz}{1-z}\ \Phi\left(\sum_{n\geqslant 0}\left(\frac{xz}{1-z}\right)^n\right).$$

We can summarize this with the relation

$$F(x,z) = 1 + x\cdot h(z) F(x, h(z)),$$

where $h(z) = z/(1-z)$ is the Möbius transformation associated to the matrix $\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$.

The compositions of $h(z)$ are $h^{\circ k}(z) = z/(1-kz)$ and for any integer m, we have

$$F(x,z) = \sum_{k=0}^{m-1} x^k h(z) h^{\circ 2}(z) \cdots h^{\circ k}(z) + x^m h(z) h^{\circ 2}(z) \cdots h^{\circ m}(z) F(x, h^{\circ m}(z)).$$

Since $h^{\circ m}(z) \equiv z \bmod m\mathbb{Z}[[z]]$, we get (modulo $m\mathbb{Z}[x][[z]]$)

$$\begin{aligned} F(x,z) \equiv & \sum_{k=0}^{m-1} x^{m-1-k} h(z) h^{\circ 2}(z) \cdots h^{\circ(m-1-k)}(z) \\ & + x^m h(z) h^{\circ 2}(z) \cdots h^{\circ(m-1)}(z) z F(x,z), \end{aligned}$$

hence $(1-z)(1-2z)\cdots(1-(m-1)z)F(x,z)$ is congruent (modulo $m\mathbb{Z}[x][[z]]$) to

$$\sum_{k=0}^{m-1} [(xz)^{m-1-k}(1+kz)(1+(k-1)z)\cdots(1+z)] + (xz)^m F(x,z).$$

Given a prime p, we can imbed the situation in the p-adic ring $\mathbb{Z}_p$ and the above congruence is still valid modulo $m\mathbb{Z}_p[x][[z]]$. The particular case when m is divisible by p, say $m = np$, is interesting: in [2], we proved that $(x)_{np} \equiv (x^p - x)^n \bmod (np/2)\mathbb{Z}_p[x]$, and taking

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