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Hanoi graphs and some classical numbers

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Abstract

The Hanoi graphs H_p^n model the p-pegs n-discs Tower of Hanoi problem(s). It was previously known that Stirling numbers of the second kind and Stern's diatomic sequence appear naturally in the graphs H_p^n. In this note, second-order Eulerian numbers and Lah numbers are added to this list. Considering a variant of the p-pegs n-discs problem, Catalan numbers are also encountered.

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1. Introduction

All the time since the French number theorist Lucas invented the Tower of Hanoi (TH) in 1883, the puzzle has presented a challenge in mathematics as well as in computer science and psychology. The classical problem consists of three pegs and is by now well-understood (cf. [5,7] and references therein). On the other hand, as soon as there are at least four pegs, the problem turns into a notorious open question (see [1,10,12,21] for recent results).

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The TH problem consists of $p \geqslant 3$ pegs and $n \geqslant 1$ discs of different sizes. A *legal move* is a transfer of the topmost disc from one peg to another peg such that no disc is moved onto a smaller one. Initially, all discs lie on one peg in small-on-large ordering, that is, in a *perfect state*. The objective is to transfer all the discs from one perfect state to another in the minimum number of legal moves. A state (= distribution of discs on pegs) is called *regular* if on every peg the discs lie in the small-on-large ordering.

The natural mathematical object to model the TH problem(s) are the graphs defined as follows. Let $p \geqslant 1$, $n \geqslant 1$, then the *Hanoi graph* H_p^n has all regular states of the p-pegs n-discs problem, two states being adjacent whenever one is obtained from the other by a legal move. Note that for any $n \geqslant 1$, H_1^n is the one-vertex graph K_1. If we have two pegs, only the smallest disc can be moved in any regular state. Hence H_2^n is the disjoint union of 2^{n-1} copies of the complete graph on two vertices K_2.

The graphs H_p^n were studied from the graph theory point of view in [9] where it is proved that they are Hamiltonian and that the only planar Hanoi graphs, besides the trivial cases H_1^n and H_2^n, are H_4^1, H_4^2, and H_3^n, $n \geqslant 1$. In this note we study these graphs from the combinatorial counting point of view. More precisely, we demonstrate that many classical numbers appear naturally in the Hanoi graphs.

For a graph G let $V(G)$ be the set of its vertices and $E(G)$ the set of its edges. It is easy to see that $|V(H_p^n)| = p^n$ and it has been shown in [11] that for $p \geqslant 3$,

$$|E(H_p^n)| = \frac{1}{2}\sum_{k=1}^{p}\left(k\left(p-\frac{1}{2}\right)-\frac{1}{2}k^2\right)\left\{ {n \atop k} \right\} p^{\underline{k}},$$

where $\left\{ {n \atop k} \right\}$ denote Stirling numbers of the second kind and $p^{\underline{k}} = p(p-1)\cdots(p-k+1)$. Moreover, it is straightforward to verify that the formula holds for $p=1$ and $p=2$ as well. So Stirling numbers of the second kind present the first instance of classical numbers that appear in Hanoi graphs.

In the next section we count some vertex subsets of Hanoi graphs and among others detect the second-order Euler numbers and the Lah numbers. Then, in Section 3, we consider the n-in-a-row TH problem and relate it with the Catalan numbers. We conclude by recalling that Stern's diatomic sequence appears in the Hanoi graphs as well.

2. Vertex subsets of Hanoi graphs

Consider the TH problem with $p \geqslant 2$ pegs numbered $1, 2, \ldots, p$ and $n \geqslant 1$ discs numbered $1, 2, \ldots, n$. We assume that the discs are ordered by size, disc 1 being the smallest one. Suppose that on peg i of a regular state we have discs $j, j+1, \ldots, j+k$ for some $j \geqslant 1$, and $k \geqslant 0$. (Of course, $j+k \leqslant n$.) Then we say that the discs $j, j+1, \ldots, j+k$ form a *superdisc* (on peg i). In addition, we also consider an empty peg as an (empty) superdisc. We call a regular state of the TH problem a *superdisc state* if the discs on every peg form a superdisc. A vertex of H_p^n that corresponds to a superdisc state will be called a *superdisc vertex*.

The term "superdisc" has been coined by Hinz in [6]. In the same paper, the term "presumed minimum solution" was also invented to describe proposed solutions for the