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On the Krein and Friedrichs extensions of a positive Jacobi operator

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Abstract

We show that for a positive linear operator acting in ℓ^2 and defined from

$$a_n x_{n+1} + b_n x_n + a_{n-1} x_{n-1}$$

its so-called Friedrichs and Krein extensions may be explicitly characterized by boundary conditions as $n \to \infty$.

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1. Introduction

We consider an infinite Jacobi matrix

$$J = \begin{pmatrix} b_0 & a_0 & 0 & 0 & 0 & \dots \\ a_0 & b_1 & a_1 & 0 & 0 & \dots \\ 0 & a_1 & b_2 & a_2 & 0 & \dots \\ 0 & 0 & a_2 & b_3 & a_3 & \\ \vdots & \vdots & & \ddots & \ddots & \ddots \end{pmatrix},$$

where $a_n > 0$ and $b_n \in \mathbb{R}$ for $n \geqslant 0$. Given a sequence $x = (x_n)$ of complex numbers, Jx is again a sequence of complex numbers. If we set $a_{-1} = 0$,

$$(Jx)_n = a_{n-1}x_{n-1} + b_n x_n + a_n x_{n+1}, \quad n \geqslant 0.$$

In order to define operators from the matrix J, we introduce the Hilbert space ℓ^2 of complex sequences $x = (x_n)$ with $\sum |x_n|^2 < \infty$. As usual, we denote by $(\cdot, \cdot)$ the inner product

$$(x, y) = \sum_0^\infty x_n \overline{y_n}, \quad x, y \in \ell^2.$$

The maximal operator $T_{\max}$ is defined by

$$(T_{\max} x)_n = (Jx)_n, \quad n \geqslant 0$$

on the domain

$$D := D(T_{\max}) = \{x \in \ell^2 : Jx \in \ell^2\}.$$

The minimal operator $T_{\min}$ is the closure (in ℓ^2) of the so-called preminimal operator T which is the restriction of $T_{\max}$ to the domain

$$D(T) = \{x \in \ell^2 : x_n = 0 \text{ for all but a finite number of values of } n\}.$$

It is straightforward to see that $T_{\min}$ is a densely defined symmetric operator and that

$$T^*_{\min} = T_{\max}, \quad T^*_{\max} = \overline{T} = T_{\min}.$$

As is well-known, the theory of Jacobi matrices is strongly connected to the theory of orthogonal polynomials and the moment problem on the real line (also called the Hamburger moment problem), see e.g. [1]. In particular, $T_{\min}$ is self-adjoint if and only if the corresponding moment problem is determinate, i.e. has a unique solution. When μ is a measure on the real line, the nth moment of μ is given by

$$s_n = \int_{\mathbb{R}} x^n \, \mathrm{d}\mu(x),$$

provided that the integral exists. The moment problem consists of deciding which sequences $(s_n)_{n\geqslant 0}$ of real numbers are moment sequences and to which extent a positive measure is