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Absolute-valued algebras with involution, and infinite-dimensional Terekhin's trigonometric algebras ☆

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Abstract

We prove that, if A is an absolute-valued $*$-algebra in the sense of [K. Urbanik, Absolute valued algebras with an involution, Fund. Math. 49 (1961) 247–258], then the normed space of A becomes a trigonometric algebra (in the meaning of [P.A. Terekhin, Trigonometric algebras, J. Math. Sci. (New York) 95 (1999) 2156–2160]) under the product $\wedge$ defined by $x \wedge y := (x^* y - y^* x)/2$. Moreover, we show that, "essentially," all infinite-dimensional complete trigonometric algebras derive from absolute-valued $*$-algebras by the above construction method.

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1. Introduction

Given nonzero elements x, y of a real pre-Hilbert space, we define as usual the angle $\alpha := \alpha(x, y)$ between x and y by the equality $\cos\alpha := (x \mid y)/\|x\|\|y\|$. By a *trigonometric algebra* we mean a nonzero real pre-Hilbert space B endowed with a (bilinear) product $\wedge : B \times B \to B$ satisfying

$$\|x \wedge y\| = \|x\|\|y\| \sin\alpha$$

for all $x, y \in B \setminus \{0\}$. We note that the above requirement is equivalent to

$$\|x \wedge y\|^2 + (x \mid y)^2 = \|x\|^2\|y\|^2.$$

The motivating example for trigonometric algebras is the Euclidean three-dimensional space endowed with the usual vector product. Since for every x in a trigonometric algebra we have $x \wedge x = 0$, trigonometric algebras are anticommutative.

Trigonometric algebras have been introduced recently by P.A. Terekhin [7], who shows that *the dimensions of finite-dimensional trigonometric algebras are precisely* 1, 2, 3, 4, 7, *and* 8. The existence of complete trigonometric algebras of arbitrary infinite Hilbertian dimension is implicitly known in [4]. Indeed, we have the following.

Example 1.1. Let H be any infinite-dimensional real Hilbert space. Take an orthonormal basis U of H, together with an injective mapping $\vartheta : U \times U \to U$. Then the mapping $(u, v) \to (\vartheta(u, v) - \vartheta(v, u))/\sqrt{2}$, from $U \times U$ to H, extends to a product $\wedge$ on H converting H into a trigonometric algebra (see Remark 1.6 of [4] for details).

The aim of the present paper is to entering the structure of infinite-dimensional trigonometric algebras, by relating them to the so called "absolute-valued $*$-algebras." An *absolute value* on a real or complex algebra A is a norm $\|\cdot\|$ on the vector space of A satisfying

$$\|xy\| = \|x\|\|y\|$$

for all $x, y \in A$. By an *absolute-valued algebra* we mean a nonzero real or complex algebra endowed with an absolute value. *Absolute-valued $*$-algebras* are defined as those absolute-valued real algebras A endowed with an isometric algebra involution $*$ which is different from the identity operator and satisfies $xx^* = x^*x$ for every $x \in A$. Absolute-valued $*$-algebras were introduced in the early paper of K. Urbanik [8], and have been reconsidered by B. Gleichgewicht [3], Urbanik himself [9], M.L. El-Mallah [1,2], and A. Rochdi [5]. The reader is referred to the recent survey paper [6] for a complete view of the theory of absolute-valued algebras.

To precisely reviewing our results, let us introduce some additional definitions. By a *super-trigonometric algebra* we mean a nonzero real pre-Hilbert space B endowed with a product $\wedge : B \times B \to B$ satisfying

$$(x \wedge y \mid u \wedge v) = (x \mid u)(y \mid v) - (x \mid v)(y \mid u)$$