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# Irreducible polynomials and full elasticity in rings of integer-valued polynomials

Scott T. Chapman [*,1], Barbara A. McClain [2]

*Trinity University, Department of Mathematics, One Trinity Place, San Antonio, TX 78212-7200, USA*

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**Abstract**

Let $D$ be a unique factorization domain and $S$ an infinite subset of $D$. If $f(X)$ is an element in the ring of integer-valued polynomials over $S$ with respect to $D$ (denoted $\mathrm{Int}(S, D)$), then we characterize the irreducible elements of $\mathrm{Int}(S, D)$ in terms of the fixed-divisor of $f(X)$. The characterization allows us to show that every nonzero rational number $n/m$ is the leading coefficient of infinitely many irreducible polynomials in the ring $\mathrm{Int}(\mathbb{Z}) = \mathrm{Int}(\mathbb{Z}, \mathbb{Z})$. Further use of the characterization leads to an analysis of the particular factorization properties of such integer-valued polynomial rings. In the case where $D = \mathbb{Z}$, we are able to show that every rational number greater than 1 serves as the elasticity of some polynomial in $\mathrm{Int}(S, \mathbb{Z})$ (i.e., $\mathrm{Int}(S, \mathbb{Z})$ is fully elastic).


*Keywords:* Integer-valued polynomial; Irreducible element; Elasticity of factorization

* Corresponding author.
*E-mail addresses:* schapman@trinity.edu (S.T. Chapman), bmcclain@math.unl.edu (B.A. McClain).

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[2] Current address: University of Nebraska at Lincoln, Department of Mathematics, 203 Avery Hall, Lincoln, Nebraska 68588-0130, USA. Part of this work is contained in the author's Senior Honors Thesis at Trinity University.

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## 1. Introduction

A great deal of recent literature has been devoted to the study of integral domains and monoids where factorization of elements into irreducible elements is not unique. Given an integral domain (or more generally a commutative cancellative monoid) $D$, let $\mathcal{I}(D)$ represent the set of irreducible elements of $D, \mathcal{U}(D)$ the set of units of $D$ and $D^\bullet = D - \{0\}$ the multiplicative monoid of $D$. We say that $D$ is *atomic* if every element of $D^\bullet$ can be written as a product of elements from $\mathcal{I}(D)$. Given a nonzero nonunit $x$ of $D$, the principal object of interest in the study of non-unique factorizations is

$$\mathcal{L}(x) = \left\{n \mid \exists \alpha_1, \ldots, \alpha_n \in \mathcal{I}(D) \text{ with } x = \alpha_1 \cdots \alpha_n\right\}$$

or the *set of lengths of* $x$. If $D$ is atomic and $|\mathcal{L}(x)| < \infty$ for all $x \in D^\bullet$, then $D$ is called a *bounded factorization domain* (BFD). Given a BFD $D$ and $x \in D^\bullet$ with $\mathcal{L}(x) = \{n_1, \ldots, n_t\}$ where $n_i \leqslant n_{i+1}$ for $1 \leqslant i \leqslant t-1$, set

$$\rho(x) = \frac{\max \mathcal{L}(x)}{\min \mathcal{L}(x)} = \frac{n_t}{n_1}.$$

While $\rho(x)$ describes the local character of non-unique factorizations, this function can be extended to the global descriptor

$$\rho(D) = \sup\left\{\rho(x) \mid x \in D^\bullet\right\}.$$

The value $\rho(x)$ is known as the *elasticity* of $x$ and $\rho(D)$ as the *elasticity* of $D$. An extensive amount of literature is devoted to the study of elasticity (see [1] for a survey). In particular, if $D$ is the ring of integers in an algebraic number field, then the elasticity of $D$ can be bounded above using the class number [10]. Moreover, an algorithm exists for computing the elasticity of any Krull monoid with finite divisor class group [7].

This paper continues the study begun in [2,5] (which is summarized in [6]) concerning factorization properties of rings of integer-valued polynomials. If $D$ is an integral domain with quotient field $K$ and $S$ is a subset of $D$, then the ring of integer-valued polynomials over $D$ with respect to $S$ is defined by

$$\mathrm{Int}(S, D) = \left\{f(X) \mid f(X) \in K[X] \text{ with } f(s) \in D \text{ for all } s \in S\right\}$$

(if $S = D$, then we use the notation $\mathrm{Int}(D, D) = \mathrm{Int}(D)$). In [2, Proposition 1.7], it is shown that if $D$ is an integral domain and $S$ an infinite subset of $D$ such that

(1) $\mathrm{Int}(S, D)$ is atomic;
(2) there exists a discrete valuation $v$ on $K$;
(3) there exists a principal prime ideal $\mathfrak{M}$ in $D$ with $|D/\mathfrak{M}| < \infty$,

then $\rho(\mathrm{Int}(S, D)) = \infty$. In fact, there is no known example of an atomic ring of integer-valued polynomials with finite elasticity.

Our interest in further studying factorization properties of $\mathrm{Int}(S, D)$ came from a close examination of the elasticity arguments in [2,5]. In both these papers, it is shown that