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Varieties of nilpotent elements for simple Lie algebras II: Bad primes

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1. Introduction

1.1. Let G be a simple algebraic group over an algebraically closed field k of characteristic $p > 0$ and let $\mathfrak{g}$ be the (restricted) Lie algebra of G with pth power map $[p]$. The maximal ideal spectrum of the cohomology ring of the restricted enveloping algebra $\operatorname{Maxspec}(\mathrm{H}^{2\bullet}(u(\mathfrak{g}), k))$ can be identified with the variety $\mathcal{N}_1(\mathfrak{g}) = \{x \in \mathfrak{g}\colon x^{[p]} = 0\}$. When the characteristic of the field is a good prime, this variety was first described as the closure of a certain Richardson orbit by Carlson, Lin, Nakano and Parshall [4]. Their methods used the techniques developed by Nakano, Parshall and Vella [23] which involved the verification of a conjecture of Jantzen on the support varieties of Weyl modules. More recently, in [30] the authors investigated a more general question. Let ρ be a finite-dimensional representation of $\mathfrak{g}$ which is realized as the derivative of a representation of G. For $r \geqslant 0$, set

$$\mathcal{N}_{r,\rho}(\mathfrak{g}) = \left\{x \in \mathcal{N}(\mathfrak{g})\colon \rho(x)^r = 0\right\},$$

where $\mathcal{N}(\mathfrak{g})$ is the variety of nilpotent elements of $\mathfrak{g}$. The variety $\mathcal{N}_{r,\rho}(\mathfrak{g})$ is a G-invariant subvariety of $\mathcal{N}(\mathfrak{g})$. Since there are finitely many G-orbits on $\mathcal{N}(\mathfrak{g})$, one can ask, how can

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this variety can be expressed as a finite union of orbit closures. The authors determined $\mathcal{N}_{r,\rho}(\mathfrak{g})$ when ρ is either a minimal dimensional irreducible representation or the adjoint representation of $\mathfrak{g}$ and the characteristic of the field k is a good prime relative to the underlying root system of $\mathfrak{g}$. The methods used did not involve the machinery in [23] and one can easily recover all the calculations in [4] by setting $r = p$.

One of the main objectives of this paper is to determine the restricted nullcone $\mathcal{N}_1(\mathfrak{g})$ over fields of bad characteristic. Since $\mathcal{N}_1(\mathfrak{g}) \cong \mathrm{Maxspec}(\mathrm{H}^{2\bullet}(u(\mathfrak{g}), k))$, the determination of $\mathcal{N}_1(\mathfrak{g})$ for bad primes is a key step in the study of support varieties for restricted Lie algebras. We have recently used our determination of $\mathcal{N}_1(\mathfrak{g})$ to calculate support varieties of Weyl modules over fields of bad characteristic [31]. Our results should also be useful in the investigation of support varieties of nonrestricted representations for $\mathfrak{g}$ (see [10, Theorem 4.2]).

We will invoke the general strategy that was established in [30] by providing descriptions of the varieties of both unipotent and nilpotent elements of order r in $\mathfrak{g}$ when the characteristic of the underlying field is a bad prime. Let ρ be a finite-dimensional representation of G and let $\mathcal{U}(G)$ be the variety of unipotent elements in G. One can define a subvariety $\mathcal{U}_{r,\rho}(G)$ of $\mathcal{U}(G)$, which is analogous to $\mathcal{N}_{r,\rho}(\mathfrak{g})$, by letting

$$\mathcal{U}_{r,\rho}(G) = \bigl\{x \in \mathcal{U}(G)\colon\ \bigl(\rho(x) - 1\bigr)^r = 0\bigr\}.$$

An analog of the restricted nullcone for the unipotent variety would be $\mathcal{U}_1(G) = \{x \in \mathcal{U}(G)\colon\ x^p = 1\}$. We will refer to this as the *restricted unipotent variety*. Since there are finitely many unipotent classes it is also reasonable to try to describe $\mathcal{U}_{r,\rho}(G)$ as a finite union of closures of such classes.

For good primes, there exists a G-equivariant isomorphism between $\mathcal{U}(G)$ and $\mathcal{N}(\mathfrak{g})$. McNinch [19] has shown that, under mild hypotheses on G, the Bardsley–Richardson isomorphism restricts to give a G-equivariant isomorphism between $\mathcal{U}_1(G)$ and $\mathcal{N}_1(\mathfrak{g})$. In particular, there are bijective correspondences between nilpotent orbits in $\mathcal{N}(\mathfrak{g})$ (respectively $\mathcal{N}_1(\mathfrak{g})$) and unipotent classes in $\mathcal{U}(G)$ (respectively $\mathcal{U}_1(G)$) over fields of good characteristic.

From our computations of $\mathcal{N}_{r,\rho}(\mathfrak{g})$ and $\mathcal{U}_{r,\rho}(G)$, we set $r = p$ to obtain concrete descriptions of the restricted nullcone $\mathcal{N}_1(\mathfrak{g})$ and the restricted unipotent variety $\mathcal{U}_1(G)$ for bad primes. Our results in conjunction with work in [4] demonstrate that these varieties are indeed irreducible for all primes, thus answering an old question posed by Friedlander and Parshall [9, (3.4)].

Furthermore, in this process, we establish a remarkable fact: there is an order preserving bijection between the nilpotent orbits in $\mathcal{N}_1(\mathfrak{g})$ and unipotent classes in $\mathcal{U}_1(G)$ for all primes. This is quite surprising because for bad primes the total number of nilpotent orbits in $\mathcal{N}(\mathfrak{g})$ and unipotent classes in $\mathcal{U}(G)$ are often different (see [5, §5.11]). Computational methods were essential for obtaining these results.

The paper is organized as follows. After the notation for the paper is introduced, we look at $\mathcal{N}_{r,\rho}(\mathfrak{g})$ and $\mathcal{U}_{r,\rho}(G)$ where ρ is the standard representation for classical simple groups G in Section 2. The results in this section will use early work of Großer [11], Hesselink [12] and Spaltenstein [25,26] who determined the nilpotent orbits and unipotent classes and their closure orderings in this setting. In Section 3, we determine $\mathcal{U}_{r,\rho}(G)$