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Affine pseudo-coverings of algebraic surfaces ☆

Masayoshi Miyanishi

School of Science and Technology, Kwansei Gakuin University, 2-1 Gakuen, Sanda, Hyogo 669–1337, Japan

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Abstract

An affine pseudo-covering $f : Y \to X$ of smooth affine varieties is an étale morphism whose image contains all codimension one points of X. If f splits an étale endomorphism $\varphi : X \to X$ as $\varphi = f \cdot g$ with a dominant morphism $g : X \to Y$, then f and g are affine pseudo-coverings under some additional conditions which are satisfied when X is the affine n-space $\mathbb{A}^n$. Motivated by the Jacobian problem, we consider an affine pseudo-coverings in the case where Y or X is isomorphic to the affine plane $\mathbb{A}^2$.

Keywords: Affine pseudo-plane; Cyclic and Platonic $\mathbb{A}^1$-fiber spaces; ML_i surface; Affine pseudo-coverings; Generalized Jacobian Conjecture

0. Introduction

We are motivated by the following problem.

Generalized Jacobian Conjecture. *Let X be a normal algebraic variety defined over the complex field $\mathbb{C}$ and let $\varphi : X \to X$ be an étale endomorphism. Then φ is a finite morphism.*

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E-mail address: miyanisi@ksc.kwansei.ac.jp.

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If we take X to be the affine space $\mathbb{A}^n$, it is the Jacobian conjecture. In order to avoid the confusion of the source X and the target X of the morphism φ, we denote it by $\varphi : X_1 \to X_2$, where $X_1 = X_2 = X$. Suppose that there exists a non-isomorphic, étale endomorphism φ of X. We are interested in normal algebraic varieties Y which might factorize φ in the sense that $\varphi : X_1 \xrightarrow{\varphi_1} Y \xrightarrow{\varphi_2} X_2$ and that φ_1 and φ_2 are étale. We may assume that they are almost surjective in the sense that the image includes all codimension one points. In general, given a morphism $f : X \to Y$ of normal affine varieties, we say that X is an *affine pseudo-covering* of Y if f is almost surjective and étale.

Let $\varphi : X \to X$ be a surjective étale endomorphism of an affine variety which is not an automorphism. There are, in fact, such examples of affine algebraic surfaces [7]. Taking an arbitrarily high iteration of φ, we have an affine pseudo-covering of arbitrarily high degree of X. If X is isomorphic to $\mathbb{A}^n$, we are interested in what kind of algebraic varieties are affine pseudo-coverings of $\mathbb{A}^n$ or what kind of algebraic varieties have $\mathbb{A}^n$ as affine pseudo-coverings. These points of view are the main motivations of the present article. Since we need structure theorems on algebraic varieties, we consider these problems mostly in the case of algebraic surfaces.

If a smooth affine algebraic surface X has $\mathbb{A}^2$ as an affine pseudo-covering, then $\pi_1(X)$ is a finite group and X has log Kodaira dimension $-\infty$. Since X has then an $\mathbb{A}^1$-fibration over the affine line $\mathbb{A}^1$ or the projective line $\mathbb{P}^1$, the condition $\pi_1(X)$ be finite entails the data on multiplicities of multiple fibers of the $\mathbb{A}^1$-fibration. We shall list up all possibilities of X in the first section and look into the structures of these surfaces. The Makar-Limanov invariant or the concept of ML_i surface with $i = 0, 1, 2$ derived from the invariant plays a supplementary role in classifying these surfaces. Important classes of surfaces constitute of affine pseudo-planes, cyclic $\mathbb{A}^1$-fiber spaces over $\mathbb{P}^1$ and Platonic $\mathbb{A}^1$-fiber spaces over $\mathbb{P}^1$. In the second section, we shall consider affine algebraic surfaces which have affine pseudo-planes as covers of affine pseudo-coverings. In the third section, we consider the existence of affine pseudo-coverings $f : X \to \mathbb{A}^2$. In regards to the Jacobian conjecture, it is expected that a smooth affine surface which has $\mathbb{A}^2$ as a cover of an affine pseudo-covering is not a cover of an affine pseudo-covering of $\mathbb{A}^2$.

Some related observations are made in Wright [14] such as the non-existence of étale morphisms from certain affine algebraic surfaces to the affine plane.

Throughout the article, the ground field is assumed to be the complex field $\mathbb{C}$. A curve C on a smooth projective surface is called an (n)-curve if C is isomorphic to $\mathbb{P}^1$ and $(C^2) = n$. A (0)-curve C on a smooth projective rational surface defines a $\mathbb{P}^1$-fibration for which C is a fiber. For a smooth algebraic surface X, we denote by $\rho(X)$ the Picard number, i.e., the rank of $\mathrm{Pic}(X)$ modulo numerical equivalence. We assume that $\Gamma(X, \mathcal{O}_X)^* = \mathbb{C}^*$ unless otherwise stated explicitly. We denote by $\mathbb{A}^1_*$ the affine line with one point punctured.

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1. Affine pseudo-coverings with $\mathbb{A}^2$ as covers

Let X be a smooth affine variety and let $f : Y \to X$ be a morphism of algebraic varieties. We say that f is *almost surjective* if $\mathrm{codim}_X(X - f(Y)) \geqslant 2$ and that Y is an