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Bilinear forms on Frobenius algebras

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Abstract

We analyze the homothety types of associative bilinear forms that can occur on a Hopf algebra or on a local Frobenius k-algebra R with residue field k. If R is symmetric, then there exists a unique form on R up to homothety iff R is commutative. If R is Frobenius, then we introduce a norm based on the Nakayama automorphism of R. We show that if two forms on R are homothetic, then the norm of the unit separating them is central, and we conjecture the converse. We show that if the dimension of R is even, then the determinant of a form on R, taken in $\dot{k}/\dot{k}^2$, is an invariant for R.

Keywords: Bilinear form; Frobenius algebra; Homothety; Hopf algebra; Isometry; Local algebra; Nakayama automorphism; Ore extension; Symmetric algebra

1. Introduction

Let R be a finite-dimensional algebra over a field k. We say R is a *Frobenius algebra* if there exists a nondegenerate bilinear form $B : R \times R \to k$ that is *associative* in the sense that $B(rs, t) = B(r, st)$, $\forall r, s, t \in R$. We say R is a *symmetric algebra* if there exists a nondegenerate associative symmetric bilinear form $B : R \times R \to k$. These properties are equivalent to the existence of an isomorphism between R and its k-dual $\hat{R} := \mathrm{Hom}_k(R, k)$ as left R-modules, respectively, as (R, R)-bimodules.

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Since many different isomorphisms between R and $\hat{R}$ exist, we also have many bilinear forms. A natural question to ask is whether the various forms are *isometric*, that is, the same under change of basis. It is trivial to observe that any form may be scaled by a nonzero constant from k, so we define two forms B and B' to be *homothetic* if there exists a change of basis $V \in \mathrm{Aut}_k(R)$ and a scalar $\alpha \in \dot{k} := k - \{0\}$ such that $B'(r, s) = \alpha B(Vr, Vs)$, $\forall r, s \in R$. We then ask instead when two forms on R are homothetic.

In this paper we will study the question above in the case when R has an ideal $\mathfrak{m}$ with $R/\mathfrak{m} \simeq k$. For example, this condition is satisfied by the group algebra $R = kG$, where G is any finite group, by taking $\mathfrak{m}$ to be the kernel of the augmentation map $\epsilon(\sum \alpha_g g) = \sum \alpha_g \in k$. It is also true for Hopf algebras, whose definition includes the existence of a counit $\epsilon : R \to k$. For most of our results we will also need to assume that k has good characteristic.

We will show that in the local symmetric case, there exists a unique form on R up to homothety iff R is commutative. For Frobenius algebras that are not symmetric, we will introduce a norm based on the order of the Nakayama automorphism, a distinguished k-algebra automorphism of R that measures how far R is from being a symmetric algebra. (The automorphism is the identity iff R is symmetric.) We will show that if two forms on R are homothetic, then the norm of the unit separating them is central, and we will conjecture the converse. Finally, we will study the determinant of a form on R and show that in even dimension, the value of the determinant in $\dot{k}/\dot{k}^2$ is an invariant of the algebra.

The idea of comparing an algebra with its dual was pioneered by F.G. Frobenius himself [2] in connection with representations of finite groups, and group algebras have remained important examples of symmetric algebras. Nakayama gave new examples and developed the main properties of Frobenius algebras and symmetric algebras in [7–9]. More recently, the group algebra example was generalized when Larson and Sweedler [5] showed that all finite-dimensional Hopf algebras are Frobenius. (See [1] for a treatment of the ubiquity of Hopf algebras.) Modern interest in Frobenius algebras has grown far beyond their representation-theoretic origins as connections have been discovered to such diverse areas as topological quantum field theories, Gorenstein rings in commutative algebra, coding theory, and the Yang–Baxter equation. For an excellent reference on the subject, see [3].

2. Preliminaries and examples

Let k be a field and R be a finite-dimensional k-algebra. Throughout this paper, we will use the word *form* (respectively, *symmetric form*) to mean a nondegenerate bilinear form (respectively, nondegenerate symmetric bilinear form) $B : R \times R \to k$ that is *associative* in the sense that $B(rs, t) = B(r, st)$, $\forall r, s, t \in R$.

In [3, Theorems 3.15 and 16.54], we have:

Theorem 1. *The following conditions are equivalent*:

(1) $R \simeq \hat{R}$ *as left* R*-modules* (*respectively, as* (R, R)*-bimodules*).
(2) *There exists a linear functional* $\lambda : R \to k$ *whose kernel contains no nonzero left ideals.* (*Respectively,* $\lambda(rs) = \lambda(sr)$, $\forall r, s \in R$.)