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Fixed point subalgebras of extended affine Lie algebras

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Abstract

It is a well-known result that the fixed point subalgebra of a finite dimensional complex simple Lie algebra under a finite order automorphism is a reductive Lie algebra so it is a direct sum of finite dimensional simple Lie subalgebras and an abelian subalgebra. We consider this for the class of extended affine Lie algebras and are able to show that the fixed point subalgebra of an extended affine Lie algebra under a finite order automorphism (which satisfies certain natural properties) is a sum of extended affine Lie algebras (up to existence of some isolated root spaces), a subspace of the center and a subspace which is contained in the centralizer of the core. Moreover, we show that the core of the fixed point subalgebra modulo its center is isomorphic to the direct sum of the cores modulo centers of the involved summands.

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0. Introduction

In 1955, A. Borel and G.D. Mostow [7] proved that the fixed point subalgebra of a finite dimensional complex simple Lie algebra under a finite order automorphism is a reductive Lie algebra. A natural question which arises here is what can we say about the fixed points of a finite order automorphism of an *extended affine Lie algebra* (EALA for short). EALA's are natural generalizations of finite dimensional complex simple Lie algebras and affine Kac–Moody Lie algebras. They are axiomatically defined (see Definition 1.5) and the axioms guarantee the existence of analogues of Cartan subalgebras, root systems, invariant forms, etc. A root of an EALA is called *isotropic* if it is orthogonal to itself, with respect to the form. The dimension of the real span of the isotropic roots is called the *nullity* of the Lie algebra. A finite dimensional simple Lie algebra is an EALA of nullity zero, and a tame EALA is an affine Lie algebra if and only if its nullity is 1 (see [1] for details). Thus EALAs form a natural class of algebras in which to consider extensions of the result of Borel and Mostow.

Here we would like to explain a procedure which has been the most general theme of constructing affine Lie algebras and their generalizations, since the birth of Kac–Moody Lie algebras in 1968.

Let $(\mathcal{G}, (\cdot,\cdot), \mathcal{H})$ be an EALA with root system R (in particular, $\mathcal{G}$ can be a finite dimensional simple Lie algebra or an affine Lie algebra). Let σ be a finite order automorphism of $\mathcal{G}$ which stabilizes $\mathcal{H}$ and leaves the form invariant. Assume also that the fixed point subalgebra of $\mathcal{H}$ (with respect to σ) is self-centralizing in the fixed point subalgebra of $\mathcal{G}$. Consider the Lie algebra

$$\mathrm{Aff}(\mathcal{G}) := \big(\mathcal{G} \otimes \mathbb{C}\big[t, t^{-1}\big]\big) \oplus \mathbb{C}c \oplus \mathbb{C}d,$$

where c is central, $d = t\frac{d}{dt}$ is the degree derivation so that $[d, x \otimes t^n] = nx \otimes t^n$, and multiplication is given by

$$\big[x \otimes t^n, y \otimes t^m\big] = [x, y] \otimes t^{n+m} + n(x, y)\delta_{m+n,0}c.$$

Extend the form $(\cdot,\cdot)$ to $\mathrm{Aff}(\mathcal{G})$ so that c and d are naturally paired. Set $\widetilde{\mathcal{H}} = \mathcal{H} \oplus \mathbb{C}c \oplus \mathbb{C}d$. Then the triple

$$\big(\mathrm{Aff}(\mathcal{G}), (\cdot,\cdot), \widetilde{\mathcal{H}}\big)$$

is again an EALA with root system $\widetilde{R} = R + \mathbb{Z}\delta$ where δ is the linear functional on $\widetilde{\mathcal{H}}$ defined by $\delta(d) = 1$ and $\delta(\mathcal{H} \oplus \mathbb{C}c) = 0$. Extend σ to an automorphism of $\mathrm{Aff}(\mathcal{G})$ by

$$\sigma\big(x \otimes t^i + rc + sd\big) = \zeta^{-i}\sigma(x) \otimes t^i + rc + sd,$$

where $\zeta = e^{2\pi\sqrt{-1}/m}$ and $\sigma^m = \mathrm{id}$. Let $\mathrm{Aff}(\mathcal{G})^\sigma$ be the fixed points of σ and $\widetilde{\mathcal{H}}^\sigma$ be the fixed pints of $\widetilde{\mathcal{H}}$ under σ. It follows that $\mathrm{Aff}(\mathcal{G})^\sigma$ has a root space decomposition with respect to $\widetilde{\mathcal{H}}^\sigma$ such that if the corresponding root system has some nonisotropic roots, then

$$\big(\mathrm{Aff}(\mathcal{G})^\sigma, (\cdot,\cdot), \widetilde{\mathcal{H}}^\sigma\big)$$