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Conjugacy separability of certain Seifert 3-manifold groups

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Abstract

It is well known that Seifert 3-manifold groups are residually finite. Niblo [J. Pure Appl. Algebra 78 (1992) 77–84] showed that they are double coset separable. Applying this result we show that, except for some special cases, most of the Seifert 3-manifold groups are conjugacy separable.

Keywords: Seifert 3-manifold groups; Generalized free products; Residually finite; Conjugacy separable; Subgroup separable; Double coset separable

1. Introduction

Let S be a subset of a group G. Then G is said to be *S-separable* if, for each $x \in G \backslash S$, there exists a normal subgroup N_x of finite index in G such that $x \notin N_x S$. Equivalently

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S is a closed subset of G in the profinite topology of G. If $S = \{1\}$, then G is *residually finite* ($\mathcal{RF}$). If for each $x \in G$, G is $\{x\}^G$-separable, where $\{x\}^G$ is the conjugacy class of x in G, then G is called *conjugacy separable* (c.s.). Residual and separability properties are of interest to both group theorists and topologists. They are related to the solvability of the word problem, conjugacy problem and generalized word problem (Mal'cev [11] and Mostowski [12]). Topologically they are related to problems on the embeddability of equivariant subspaces in their regular covering spaces (Scott [16], Niblo [13]).

Seifert 3-manifold groups have nice residual and separability properties. It is well known that they are residually finite (Hempel [7]). In general it is not known which 3-manifold groups are residually finite. Thurston [22] proved that fundamental groups of Haken manifolds are residually finite. Scott [16] showed that Fuchsian groups and Seifert 3-manifold groups are subgroup separable (or Lerf). The results were improved by Niblo [13] who showed that these groups are double coset separable. In [5], Fine and Rosenberger showed that Fuchsian groups are conjugacy separable. It is natural to ask whether Seifert 3-manifold groups are conjugacy separable. In this paper we show that most of the Seifert 3-manifold groups are conjugacy separable. Formanek [6] showed that polycyclic groups are conjugacy separable. Ribes, Segal and Zalesskii [15] showed that generalized free products of polycyclic groups amalgamating cyclic subgroups are conjugacy separable. Applying this result it is easy to show that Seifert 3-manifold groups with non-empty boundary are conjugacy separable. Using this, in [3], Allenby, Kim and Tang proved that the outer automorphism groups of these groups are residually finite.

There are a number of other results on conjugacy separability. In particular, Stebe proved the following results:

(1) Free products of conjugacy separable groups are conjugacy separable [17];
(2) The group obtained by adjoining a root to a free group is conjugacy separable [18];
(3) Groups of hose knots are conjugacy separable [19];
(4) The linear groups $\mathrm{GL}_2(\mathbb{Z})$, $\mathrm{SL}_2(\mathbb{Z})$, $\mathrm{GL}_n(\mathbb{Z}_p)$, $\mathrm{SL}_n(\mathbb{Z}_p)$ for all positive integers n and primes p are conjugacy separable [20].

Tang [21] showed that generalized free products of surface groups amalgamating a cyclic subgroup are conjugacy separable. Wilson and Zalesskii [23] showed that the Bianchi groups $\mathrm{PSL}_2(\mathcal{O}_d)$, where $\mathcal{O}_d$ is the ring of integers of $\mathbb{Q}(\sqrt{-d})$ for $d = 1, 2, 7, 11$, are conjugacy separable.

In Section 2, we list the terms and notation which we shall be using. We also state the known results we use frequently. In Section 3, we prove a criterion for generalized free products, amalgamating a direct product of cycles, to be conjugacy separable. In Section 4, we discuss the conjugacy separability of $G_1(s,r)$ for $s \geqslant 1$. In Section 5, we discuss the conjugacy separability of $G_1(0,r)$. We conclude that except for $G_1(0,3)$ and $G_1(1,1)$, all the $G_1(s,r)$ are conjugacy separable. We believe similar results hold for $G_2(s,r)$. We conjecture that all Seifert 3-manifold groups are conjugacy separable.