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Strong lifting

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Abstract

If I is an ideal of a ring R, we say that idempotents lift strongly modulo I if, whenever $a^2 - a \in I$, there exists $e^2 = e \in aR$ (equivalently $e^2 = e \in Ra$) such that $e - a \in I$. The higher socles of R all enjoy this property, as does the Jacobson radical J if idempotents lift modulo J. Many of the useful, basic properties of lifting modulo J are shown to extend to any ideal I with strong lifting, and analogs of the semiperfect and semiregular rings are studied. A number of examples are given that limit possible extensions of the results.

Keywords: Lifting idempotents; Socles; Semiregular rings; Semiperfect rings; Goldie torsion ideal

One of the most used general methods for determining the structure of a ring R is to first determine the structure of an image ring R/I where I is an ideal of R, and then "lift" this structure to R. This often comes down to "lifting idempotents" because many structural features of a ring are described in terms of idempotents. The most useful choice of the ideal in this process has been the Jacobson radical $J = J(R)$. One reason for the popularity of J is that, if idempotents lift modulo J, then they lift in the following stronger sense: If I is an ideal of a ring R, we say that idempotents can be lifted strongly modulo I if, whenever $a^2 - a \in I$, there exists $e^2 = e \in aR$ such that $e - a \in I$. This concept turns

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out to be left-right symmetric even if I is a one-sided ideal. The notion of strong lifting seems to be the natural requirement for ideals other than J, and the most useful properties of lifting modulo J extend. Among others, the following theorems are proved:

Theorem. *Idempotents lift strongly modulo all the higher socles of a ring.*

Theorem. *A ring R is an exchange ring if and only if idempotents lift strongly modulo every right* (*respectively left*) *ideal of R.*

If I is an ideal of R, the ring R will be called I-semiperfect (respectively I-semiregular) if R/I is semisimple (respectively regular) and idempotents lift strongly modulo I. We say that I respects a right ideal T if $T = eR \oplus S$ where $e^2 = e$ and $S \subseteq I$, and we prove:

Theorem. *R is I-semiregular if and only if I respects every principal* (*respectively finitely generated*) *right ideal of R.*

Theorem. *R is I-semiperfect if and only if I respects every right ideal of R.*

Theorem. *R is I-semiperfect if and only if*: (1) *R contains no infinite set of orthogonal idempotents outside I*; *and* (2) *every right ideal not contained in I contains an idempotent not in I.*

We also show that I-semiregularity and I-semiperfectness are inherited by related rings in a natural way; in fact, they are Morita invariants if I is the Jacobson, Baer (prime) or Levitzky radical. The paper concludes with a discussion of the case where I is the Goldie torsion right ideal Z_2^r of R:

Theorem. *R is Z_2^r-semiperfect if and only if Z_2^r respects every maximal right ideal of R*; *if and only if every nonsingular right R-module is injective.*

Throughout this paper, every ring R is associative with unity and all modules are unitary. If M is an R-module, we write $J(M)$, $\mathrm{soc}(M)$ and $Z(M)$ for the Jacobson radical, the socle, and the singular submodule of M, respectively. We write $N \subseteq^{\mathrm{ess}} M$ and $N \subseteq^{\oplus} M$ if N is an essential submodule of M, respectively a direct summand of M. When no confusion results, we abbreviate $J(R) = J$, $\mathrm{soc}(R_R) = S_r$, and $Z(R_R) = Z_r$. We write the Goldie torsion right ideal of R as Z_2^r, where $Z(R/Z_r) = Z_2^r/Z_r$, with a similar notation on the left. The left and right annihilators of a subset $X \subseteq R$ are denoted by $\mathtt{l}(X)$ and $\mathtt{r}(X)$, respectively, and we write $I \lhd R$ to indicate that I is a two-sided ideal of R. We write $\mathbb{Z}$ for the ring of integers and $\mathbb{Z}_n$ for the ring of integers modulo n. A ring, possibly with no unity is called a general ring. We denote the ring of $n \times n$ matrices over R by $M_n(R)$.