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Coefficient fields and scalar extension in positive characteristic ☆

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Abstract

Let k be a perfect field of positive characteristic, $k(t)_{\mathrm{per}}$ the perfect closure of $k(t)$ and $A = k[[X_1, \ldots, X_n]]$. We show that for any maximal ideal $\mathfrak{n}$ of $A' = k(t)_{\mathrm{per}} \otimes_k A$, the elements in $\widehat{A'_{\mathfrak{n}}}$ which are annihilated by the "Taylor" Hasse–Schmidt derivations with respect to the X_i form a coefficient field of $\widehat{A'_{\mathfrak{n}}}$.

Keywords: Complete local ring; Coefficient field; Hasse–Schmidt derivation

Introduction

Let k be a perfect field, $k_{(\infty)} = k(t)_{\mathrm{per}}$ the perfect closure of $k(t)$, $A = k[[X_1, \ldots, X_n]]$.

If k is of characteristic 0, then $k_{(\infty)} = k(t)$ and $A(t) = A \otimes_k k(t)$ is obviously noetherian. Actually, $A(t)$ is an n-dimensional regular non-local ring (see Example 2.3) whose maximal ideals have the same height ($= n$). In [8] the second author proved that there is a uniform way to obtain a coefficient field in the completions $\widehat{(A(t))_{\mathfrak{n}}}$, for all max-

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imal ideals $\mathfrak{n}$ in $A(t)$. Namely, the elements in $\widehat{(A(t))_{\mathfrak{n}}}$ which are annihilated by the partial derivatives $\frac{\partial}{\partial X_i}$ form a coefficient field of $\widehat{(A(t))_{\mathfrak{n}}}$.

In this paper, we generalize the above result to the positive characteristic case.

At first sight, in positive characteristic it seems natural to consider Hasse–Schmidt derivations instead of usual derivations (see [4, Theorem 3.17]), but Example 2.3 shows that the question is not so clear.

Consequently, in the characteristic $p > 0$ case we take the scalar extension $k \to k_{(\infty)}$ instead of $k \to k(t)$, but a new problem appears: it is not obvious that the ring $A_{(\infty)} = A \otimes_k k_{(\infty)}$ is noetherian. We have proved that result in [3].

The main result in this paper says that, for every maximal ideal $\mathfrak{n}$ in $A_{(\infty)}$, the elements in $\widehat{(A_{(\infty)})_{\mathfrak{n}}}$ which are annihilated by the "Taylor" Hasse–Schmidt derivations with respect to the X_i form a coefficient field of $\widehat{(A_{(\infty)})_{\mathfrak{n}}}$.

Let us now comment on the content of this paper.

In Section 1 we introduce our basic notations and recall some results, mainly from [3].

In Section 2 we prove our main result and give the (counter)Example 2.3.

In the appendix we give a complete proof of Normalization Lemma for power series rings over perfect fields, which is an important ingredient in the proof of Theorem 2.1 and that we have not found in the literature. Our proof closely follows the proof in [1], but the latter works only for infinite perfect fields.

1. Preliminaries and notations

All rings and algebras considered in this paper are assumed to be commutative with unit element. If B is a ring, we shall denote by $\dim(B)$ its Krull dimension and by $\Omega(B)$ the set of its maximal ideals. We shall use the letters K, L, k to denote fields and $\mathbb{F}_p$ to denote the finite field of p elements, for a prime number p. If $\mathfrak{p} \in \operatorname{Spec}(B)$, we shall denote by $\operatorname{ht}(\mathfrak{p})$ the height of $\mathfrak{p}$. Remember that a ring B is said to be *biequidimensional* if all its saturated chains of prime ideals have the same length.

If B is an integral domain, we denote by $\operatorname{Qt}(B)$ its quotient field.

If k is a ring and B is a k-algebra, the set of all derivations (respectively of all Hasse–Schmidt derivations) of B over k (cf. [5] and [6, §27]) will be denoted by $\operatorname{Der}_k(B)$ (respectively $\operatorname{HS}_k(B)$).

Now, we recall the notations and some results of [3] which are used in this paper.

For any $\mathbb{F}_p$-algebra B, we denote $B^\sharp := \bigcap_{e \geqslant 0} B^{p^e}$.

Let k be a field of characteristic $p > 0$ and consider the field extension

$$k_{(\infty)} := \bigcup_{m \geqslant 0} k\left(t^{\frac{1}{p^m}}\right) \supset k(t).$$

If k is perfect, $k_{(\infty)}$ coincides with the perfect closure of $k(t)$.