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On invariant theory of θ-groups

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Introduction

This paper is a contribution to Vinberg's theory of θ-groups, or in other words, to Invariant Theory of periodically graded semisimple Lie algebras [Vi1,Vi2]. One of our main tools is Springer's theory of regular elements of finite reflection groups [Sp], with some recent complements by Lehrer and Springer [LS1,LS2].

The ground field $\Bbbk$ is algebraically closed and of characteristic zero. Throughout, G is a connected and simply connected semisimple algebraic group, $\mathfrak{g}$ is its Lie algebra, and Φ is the Cartan–Killing form on $\mathfrak{g}$; $l = \operatorname{rk}\mathfrak{g}$.

$\operatorname{Int}\mathfrak{g}$ (respectively $\operatorname{Aut}\mathfrak{g}$) is the group of inner (respectively all) automorphisms of $\mathfrak{g}$; $\mathcal{N}$ is the nilpotent cone in $\mathfrak{g}$. For $x \in \mathfrak{g}$, $\mathfrak{z}(x)$ is the centraliser of x in $\mathfrak{g}$.

Let $\mathfrak{g} = \bigoplus_{i\in\mathbb{Z}_m} \mathfrak{g}_i$ be a periodic grading of $\mathfrak{g}$ and θ the corresponding mth order automorphism of $\mathfrak{g}$. Let G_0 denote the connected subgroup of G with Lie algebra $\mathfrak{g}_0$. Invariant Theory of θ-groups deals with orbits and invariants of G_0 acting on $\mathfrak{g}_1$. Its main result status that there is a subspace $\mathfrak{c} \subset \mathfrak{g}_1$ and a finite reflection group $W(\mathfrak{c},\theta)$ in $\mathfrak{c}$ (the *little Weyl group*) such that $\Bbbk[\mathfrak{g}_1]^{G_0} \simeq \Bbbk[\mathfrak{c}]^{W(\mathfrak{c},\theta)}$. We say that the grading is N-regular (respectively S-regular) if $\mathfrak{g}_1$ contains a regular nilpotent (respectively semisimple) element of $\mathfrak{g}$. The grading is locally free if there is $x \in \mathfrak{g}_1$ such that $\mathfrak{z}(x) \cap \mathfrak{g}_0 = \{0\}$. The same terminology also applies to θ.

In this paper, we obtain some structural results for gradings with these properties and study interrelations of these properties. Section 1 contains some preliminary material on

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θ-groups and regular elements. In Section 2, we begin with a dimension formula for semi-simple G_0-orbits in $\mathfrak{g}_1$. We also prove two "uniqueness" theorems. Recall that $\operatorname{Int}\mathfrak{g}$ is the identity component of $\operatorname{Aut}\mathfrak{g}$, and it operates on $\operatorname{Aut}\mathfrak{g}$ via conjugations. Given $m \in \mathbb{N}$, we prove that each connected component of $\operatorname{Aut}\mathfrak{g}$ contains at most one $\operatorname{Int}\mathfrak{g}$-orbit consisting of automorphisms of order m that are either N-regular or S-regular and locally free. In Section 3, we show that θ-groups corresponding to N-regular gradings enjoy a number of good properties. Let $m_1, \dots, m_l$ be the exponents of $\mathfrak{g}$ and let $\{e, h, f\}$ be a regular $\mathfrak{sl}_2$-triple such that $e \in \mathfrak{g}_1$ and $f \in \mathfrak{g}_{-1}$. For the purposes of this introduction, assume that θ is inner. Set $k_i = \#\{j \mid m_j \equiv i \pmod m\}$ ($i \in \mathbb{Z}_m$), and let ζ be a primitive mth root of unity. We prove that

(i) the eigenvalues of θ on $\mathfrak{z}(e)$ are ζ^{m_i} $(1 \leqslant i \leqslant l)$,
(ii) $\dim \mathfrak{g}_{i+1} - \dim \mathfrak{g}_i = k_{-i-1} - k_i$ for all $i \in \mathbb{Z}_m$,
(iii) the restriction homomorphism $\Bbbk[\mathfrak{g}]^G \to \Bbbk[\mathfrak{g}_1]^{G_0}$ is onto,
(iv) the G_0-action on $\mathfrak{g}_1$ admits a Kostant–Weierstrass (= KW) section.

In the general case, the definition of the k_i's becomes more involved, see Eq. (3.1), but the above assertions (ii)–(iv) remains intact.

In Section 4, it is shown that any locally free S-regular grading of $\mathfrak{g}$ is N-regular. This implies that all such gradings admit a KW-section. We also give a formula for dimension of all subspaces $\mathfrak{g}_i$ in the S-regular case. Another result is that $\dim \mathfrak{c} \leqslant k_{-1}$ for any θ-group. We then show that the G-stable cone $\pi^{-1}\pi(\mathfrak{c}) \subset \mathfrak{g}$ is a normal complete intersection. (Here $\pi : \mathfrak{g} \to \mathfrak{g}/\!\!/G$ is the quotient mapping.) In particular, if θ is S-regular or N-regular, then $\overline{G \cdot \mathfrak{g}_1}$ is a normal complete intersection. A description of the defining ideal of $\overline{G \cdot \mathfrak{g}_1}$ is also given. This material on normality relies on results of Richardson [Ri]. It is curious to note that in case $m = 2$ (i.e., θ is involutory) S-regularity is equivalent to N-regularity. But for $m > 2$ neither of these properties implies the other.

Section 5 contains a description of the coexponents for little Weyl groups, if θ is both S- and N-regular. This is based on recent results of Lehrer and Springer [LS2].

1. Vinberg's θ-groups and Springer's regular elements

Let θ be an automorphism of $\mathfrak{g}$ of finite order m. The automorphism of G induced by θ is also denoted by θ. Let ζ be a fixed primitive mth root of unity. Then θ determines a periodic grading $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}_m} \mathfrak{g}_i$, where $\mathfrak{g}_i = \{x \in \mathfrak{g} \mid \theta(x) = \zeta^i x\}$. Whenever we want to indicate the dependence of the grading on θ, we shall endow '$\mathfrak{g}_i$' with a suitable superscript. If M is a θ-stable subspace, then $M_i := M \cap \mathfrak{g}_i$. Recall some standard facts on periodic gradings (see [Vi1, Section 1]):

- $\Phi(\mathfrak{g}_i, \mathfrak{g}_j) = 0$ unless $i + j = 0$;
- Φ is non-degenerate on $\mathfrak{g}_i \oplus \mathfrak{g}_{-i}$ $(i \neq 0)$ and on $\mathfrak{g}_0$;
- in particular, $\mathfrak{g}_0$ is a reductive algebraic Lie algebra and $\dim \mathfrak{g}_i = \dim \mathfrak{g}_{-i}$;
- if $x \in \mathfrak{g}_i$ and $x = x_s + x_n$ is its Jordan decomposition, then $x_s, x_n \in \mathfrak{g}_i$.