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# Gradings on simple Jordan and Lie algebras

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**Abstract**

In this paper we describe all group gradings by a finite Abelian group $G$ of several types of simple Jordan and Lie algebras over an algebraically closed field $F$ of characteristic zero.


*Keywords:* Graded algebra; Simple Lie algebra; Involution; Simple Jordan algebra

## 1. Introduction and notation

Let $F$ be an algebraically closed field of characteristic zero, $R$ a finite-dimensional associative algebra over $F$ and $G$ a group. We say that $R$ is a $G$-graded algebra, if there is

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a vector space sum decomposition

$$R = \bigoplus_{g \in G} R_g,$$

such that

$$R_g R_h \subseteq R_{gh} \quad \text{for all } g, h \in G.$$

A subspace $V \subseteq R$ is called *graded* (or *homogeneous*) if $V = \bigoplus_{g \in G} V \cap R_g$. An element $a \in R$ is called *homogeneous* of degree $g$ if $a \in R_g$. The *support* of the $G$-grading is a subset

$$\operatorname{Supp} R = \{g \in G \mid R_g \neq 0\}.$$

An *involution* $*$ of a ring $R$ is an antiautomorphism of $R$ whose order is 2, that is, $(ab)^* = b^* a^*$ and $(a^*)^* = a$ for any $a, b \in R$. An algebra is called *involution simple* if it has no two-sided ideal different from $R$ and $\{0\}$. An element $a \in R$ is called *symmetric* if $a^* = a$ and *skew-symmetric* if $a^* = -a$. The set of $H(R, *)$ of symmetric elements is a Jordan algebra under the circle product $a \circ b = ab + ba$ (more often $a \circ b = \frac{1}{2}(ab + ba)$ and the $\frac{1}{2}$ must be defined in $R$). The set $K(R, *)$ of skew-symmetric elements is a Lie algebra under the bracket operations $[a, b] = ab - ba$. Any classical finite-dimensional simple Lie algebra over an algebraically closed field $F$ is isomorphic to a Lie algebra $K(R, *)$ of skew symmetric elements of a finite-dimensional involution simple algebra $R$, except $\mathrm{sl}(n)$, which is of the form $[K(R, *), K(R, *)]$. Similarly, any special simple finite-dimensional Jordan algebra over an algebraically closed field of $F$ characteristic different from 2, except of the algebra of non-degenerate symmetric bilinear form, is isomorphic to the set $H(R, *)$ of symmetric elements of an involution simple algebra $R$. We use these facts in this paper in order to apply our previous results [3] and [2] on the gradings of the full matrix algebras $M_n$ over an algebraically closed field $F$ of characteristic zero to the classification of the gradings by finite Abelian groups for those simple Lie and Jordan algebras which can be obtained as just above from $R = M_n$ and an appropriate involution.

An involution $*$ of a $G$-graded ring $R$ is called *graded* if, for any $g \in G$ we have $R_g^* = R_g$. We notice that the spaces $H(R, *)$ and $K(R, *)$ are $G$-graded if and only if the involution is graded and we prove (Theorem 1) that if a grading on $M_n$ admits a graded involution then the same grading can be performed by a finite Abelian group. In the case of Abelian gradings we determine (Theorem 3) all gradings for which $H(R, *)$ and $K(R, *)$, for an appropriate involution, are $G$-graded. Then we use the connection between gradings and actions by a finite Abelian group (described just below) to derive the gradings of Lie algebras of the types $B_l$, $l \geqslant 2$, $C_l$, $l \geqslant 3$, $D_l$, $l > 4$, and of the Jordan algebras $H(F_n)$ and $H(Q_n)$ (Theorems 4, 5, 6, 7). A key observation here is that any Abelian group of automorphisms of an algebra in this list extends to an Abelian group of automorphisms of the full matrix algebra of an appropriate order [7,8].

It should be noted that V. Kac [9] has essentially determined all gradings of simple Lie algebras by finite cyclic groups by previously determining the automorphisms of fi-