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Journal of Algebra 289 (2005) 70–104
JOURNAL OF Algebra
www.elsevier.com/locate/jalgebra

Hamiltonian selfdistributive quasigroups ☆

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Received 30 April 2004

Available online 3 May 2005

Communicated by Kent R. Fuller

Abstract

The problem of the existence of non-medial distributive hamiltonian quasigroups is solved. Translating this problem first to commutative Moufang loops with operators, then to ternary algebras and, finally, to cocyclic modules over $\mathbb{Z}[x, x^{-1}, (1-x)^{-1}]$, it is shown that every non-medial distributive hamiltonian quasigroup has at least 729 elements and that there are just two isomorphism classes of such quasigroups of the least cardinality. The quasigroups representing these two classes are anti-isomorphic.

Keywords: Quasigroup; Distributive; Medial; Hamiltonian

☆ The research of the first author was partially supported by the DGI and the European Regional Development Fund, jointly, through Project BFM2002-01390, and by the Comissionat per Universitats i Recerca of the Generalitat de Catalunya. The second author was supported by the institutional grant MSM 0021620839 and by the Grant Agency of Charles University, grant #269/2001/B-MAT/MFF, the third author by the Grant Agency of Czech Republic, grant #GAČR-201/02/0594.
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0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2005.03.017

0. Introduction

The first explicit allusion to the left and right instances of selfdistributivity (i.e., $(x(yz) \doteq (xy)(xz)$ and $(xy)z \doteq (xz)(yz))$ seems to appear in [41] where one can read the following comment: "*These are other cases of the distributive principle. ... These formulae, which have hitherto escaped notice, are not without interest*." Another early work [45] already contains a particular example of a (self)distributive quasigroup:

	0	1	2
0	0	2	1
1	2	1	0
2	1	0	2

This quasigroup is necessarily non-associative and plays a principal rôle in the structure theory of distributive (or, more generally, trimedial) quasigroups (see, e.g., [2,3,5,6,21,37, 50]).

The first article fully devoted to selfdistributivity is (perhaps) [11] (see also [34,51]) where, among others, normal subquasigroups are studied and an attempt is made to show that every minimal subquasigroup of a (finite) distributive quasigroup is normal (see also [16]). Actually, the latter assertion is not true. All non-medial symmetric distributive quasigroups (alias non-desarguesian planarily affine triple systems) serve as counterexamples and first constructions of these can be found in [9,18]. However, the paper [11] may be regarded as the starting point for the investigation of normality problems in distributive quasigroups.

Hamiltonian groups (i.e., (non-commutative) groups having only normal subgroups) were described (and named after W.R. Hamilton) by R. Dedekind in [14] and it was shown in [40] that a similar description takes place for hamiltonian Moufang loops, too. Furthermore, all subquasigroups of medial quasigroups (i.e., quasigroups satisfying the identity $(xy)(uv) \doteq (xu)(yv)$) are normal. That is, these quasigroups are hamiltonian. (Notice that abelian groups are included in hamiltonian structures in this paper—not usual, but technically advantageous.)

A thorough treatment (remarkable also for epic width) on cancellative distributive groupoids was written by J.-P. Soublin [50], see also [48,49]. Section IV.9 of [50] is devoted to normal subquasigroups of distributive quasigroups and, among others, it is shown that every hamiltonian symmetric (i.e., satisfying the identities $xy \doteq yx$ and $x(xy) \doteq y$) distributive quasigroup is medial. Moreover, an open problem whether there exist non-medial hamiltonian distributive quasigroups is formulated [50, p. 175]. The main aim of the present paper is to solve this problem.

In [44], it is claimed that every hamiltonian quasigroup which is either distributive or a CH-quasigroup (i.e., a symmetric quasigroup satisfying the identity $(xx)(yz) \doteq (xy)(xz)$), is medial. The proof is based on the idea that if H is a subloop of a commutative Moufang loop G and the subloop generated by H and the centre of G is normal then H is normal. However, this assertion is false, any non-associative commutative Moufang loop nilpotent of class 2 serving as an easy counterexample (in this case, every subloop containing the centre is normal and G contains a non-normal subloop). Moreover, 3.2