Available online at www.sciencedirect.com
SCIENCE DIRECT®

Journal of Algebra 289 (2005) 594–613

JOURNAL OF Algebra

www.elsevier.com/locate/jalgebra

A family of Prym–Tyurin varieties of exponent 3 ☆

H. Lange [a], S. Recillas [b,*], A.M. Rojas [c]

[a] *Mathematisches Institut, Universität Erlangen-Nürnberg, Germany*
[b] *Instituto de Matematicas, UNAM Campus Morelia and CIMAT, Guanajuato, Mexico*
[c] *Instituto de Matematicas, UNAM Campus Morelia, Mexico*

Received 1 September 2004

Available online 3 March 2005

Communicated by Corrado de Concini

Abstract

We investigate a family of correspondences associated to étale coverings of degree 3 of hyperelliptic curves. They lead to Prym–Tyurin varieties of exponent 3. We identify these varieties and derive some consequences.

Keywords: Prym–Tyurin variety; Correspondence

1. Introduction

A *correspondence* on a smooth projective curve C is by definition a divisor D on the product $C \times C$. Any correspondence D on C induces an endomorphism γ_D of the Jacobian JC. Conversely, for every endomorphism $\gamma \in \operatorname{End} C$ there is a correspondence D on C such that $\gamma = \gamma_D$. However for most correspondences D which occur in the literature, γ_D is a multiple $d \cdot 1_{JC}$ of the identity of the Jacobian. In particular, this is the case for

☆ Supported by DAAD, Conacyt 40033-F and FONDECYT No. 3040066.
* Corresponding author.
E-mail addresses: lange@mi.uni-erlangen.de (H. Lange), sevin@matmor.unam.mx (S. Recillas), anita@matmor.unam.mx (A.M. Rojas).

0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2005.01.029

any correspondence of a general curve. It were mainly these correspondences, called *of valency d*, which were studied by the classical Italian geometers (see, e.g., [8]).

At the beginning of the 1970s A. Tyurin suggested the investigation of another class of correspondences, namely effective symmetric correspondences D without fixed point on C such that γ_D satisfies an equation

$$\gamma_D^2 + (e-2)\gamma_D - (e-1) = 0.$$

For these correspondences $P = \operatorname{im}(\gamma_D - 1_{JC})$ is a *Prym–Tyurin variety* of exponent e, meaning that the restriction of the canonical polarization of JC to P is the e-fold of a principal polarization on P. Jacobians are Prym–Tyurin varieties of exponent 1, Prym varieties associated to étale double coverings are Prym–Tyurin varieties of exponent 2. On the other hand, it is not difficult to show (see [2, Corollary 12.2.4]) that every principally polarized abelian variety is a Prym–Tyurin of some high exponent. However, it seems not so easy to construct Prym–Tyurin varieties of low exponent $\geqslant 3$. First examples (associated to Fano threefolds) were investigated by Tyurin (see [9]). Other examples (associated to Weil groups of certain Lie algebras) were constructed by Kanev (see [5]).

It is the aim of this paper to study the following correspondence: let C be a hyperelliptic curve of genus $g \geqslant 3$ and $f : \widetilde{C} \to C$ an étale threefold covering. Consider the following curve in the symmetric product $\widetilde{C}^{(2)}$:

$$X = \bigl\{ p \in \widetilde{C}^{(2)} \bigm| f^{(2)}(p) \in g_2^1 \bigr\}.$$

Let ι denote the hyperelliptic involution of C and for $x \in C$ write $f^{-1}(x) = \{x_1, x_2, x_3\}$, $f^{-1}(\iota x) = \{y_1, y_2, y_3\}$ and moreover for abbreviation $P_{ij} = x_i + y_j \in \widetilde{C}^{(2)}$. Then the symmetric $(2,2)$-correspondence D on X is defined by

$$D = \bigl\{ (P_{ij}, P_{kl}) \in X \times X \bigm| i = k \text{ and } j \neq l \text{ or } i \neq k \text{ and } j = l \bigr\}.$$

We show in Section 2 under the hypothesis that X is smooth and irreducible, that $P = \operatorname{im}(\gamma_D - 1_{JX})$ is a Prym–Tyurin variety of exponent 3. In Section 3 we realise a $(2g-1)$-dimensional family of pairs (C, f) such that X is smooth and irreducible. In fact, the Galois group G of the Galois extension $Y \to \mathbb{P}^1$ of $\widetilde{C} \to \mathbb{P}^1$ is necessarily isomorphic to $S_3 \times S_3 \subset S_6$. In Section 5 we compute the dimensions of the Jacobians and Prym varieties relevant to this situation. The main result of this section is Theorem 5.3 which says that there are two trigonal curves X_1 and X_2 associated to subgroups of G such that P is canonically isomorphic as a principally polarized abelian variety to the product of Jacobians $JX_1 \times JX_2$. The trigonal covers of X_1 and X_2 have disjoint ramification locus and X is their common fibre product over $\mathbb{P}^1$. As a consequence of this and the moduli considerations of Section 4 we obtain the following consequence which seems of interest to us and for which we could not find a different proof:

Corollary (of Theorems 4.1 and 5.2). *Let X_1 and X_2 be trigonal curves with simple ramification and disjoint branching and let X denote their fibre product over $\mathbb{P}^1$ with projections $f_i : X \to X_i$. Then X is a smooth projective curve and the map $f_1^* + f_2^* : JX_1 \times JX_2 \to JX$ is an embedding.*