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Generalized MV-algebras

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Abstract

We generalize the notion of an MV-algebra in the context of residuated lattices to include non-commutative and unbounded structures. We investigate a number of their properties and prove that they can be obtained from lattice-ordered groups via a truncation construction that generalizes the Chang–Mundici Γ functor. This correspondence extends to a categorical equivalence that generalizes the ones established by D. Mundici and A. Dvurečenskij. The decidability of the equational theory of the variety of generalized MV-algebras follows from our analysis.

Keywords: Residuated lattice; MV-algebra; Lattice-ordered group; Nucleus; Categorical equivalence

1. Introduction

A *residuated lattice* is an algebra $\mathbf{L} = \langle L, \wedge, \vee, \cdot, \backslash, /, e\rangle$ such that $\langle L, \wedge, \vee\rangle$ is a lattice; $\langle L, \cdot, e\rangle$ is a monoid; and for all $x, y, z \in \mathbf{L}$,

$$x \cdot y \leqslant z \quad \Leftrightarrow \quad x \leqslant z/y \quad \Leftrightarrow \quad y \leqslant x\backslash z.$$

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Residuated lattices form a finitely based equational class of algebras (see, for example, [4]), denoted by $\mathcal{RL}$.

It is important to remark that the elimination of the requirement that a residuated lattice have a bottom element has led to the development of a surprisingly rich theory that includes the study of various important varieties of cancellative residuated lattices, such as the variety of lattice-ordered groups. See, for example, [2,4,9,12–14,18,20].

A *lattice-ordered group* (*ℓ-group*) is an algebra $\mathbf{G} = \langle G, \wedge, \vee, \cdot, {}^{-1}, e\rangle$ such that $\langle G, \wedge, \vee\rangle$ is a lattice, $\langle G, \cdot, {}^{-1}, e\rangle$ is a group, and multiplication is order preserving (or, equivalently, it distributes over the lattice operations). The variety of ℓ-groups is term equivalent to the subvariety, $\mathcal{LG}$, of residuated lattices defined by the equations $(e/x)x \approx e \approx x(x\backslash e)$; the term equivalence is given by $x^{-1} = e/x$ and $x\backslash y = x^{-1}y,\ y/x = yx^{-1}$. See [1] for an accessible introduction to the theory of ℓ-groups.

A *residuated bounded-lattice* is an algebra $\mathbf{L} = \langle L, \wedge, \vee, \cdot, \backslash, /, e, 0\rangle$ such that $\langle L, \wedge, \vee, \cdot, \backslash, /, e\rangle$ is a residuated lattice and $\mathbf{L}$ satisfies the equation $x \vee 0 \approx x$. Note that $\top = 0\backslash 0 = 0/0$ is the greatest element of such an algebra. A residuated (bounded-) lattice is called *commutative* if it satisfies the equation $xy \approx yx$ and *integral* if it satisfies $x \wedge e \approx x$.

Commutative, integral residuated bounded-lattices have been studied extensively in both algebraic and logical form, and include important classes of algebras, such as the variety of MV-algebras, which provides the algebraic setting for Łukasiewicz's infinite-valued propositional logic. Several term equivalent formulations of MV-algebras have been proposed (see, for example, [8]). Within the context of commutative, residuated bounded-lattices, MV-algebras are axiomatized by the identity $(x \to y) \to y \approx x \vee y$, which is a relativized version of the law $\neg\neg x \approx x$ of double negation; in commutative residuated lattices we write $x \to y$ for the common value of $x\backslash y$ and y/x, and $\neg x$ for $x \to 0$. The term equivalence with the standard signature is given by $x \odot y = x \cdot y$, $\neg x = x \to 0$, $x \oplus y = \neg(\neg x \cdot \neg y)$ and $x \to y = \neg x \oplus y$. The appropriate non-commutative generalization of an MV-algebra is a residuated bounded-lattice that satisfies the identities $x/(y\backslash x) \approx x \vee y \approx (x/y)\backslash x$. These algebras have recently been considered in [10,15,16] under the name pseudo-MV-algebras.

C.C. Chang proved in [7] that if $\mathbf{G} = \langle G, \wedge, \vee, \cdot, {}^{-1}, e\rangle$ is a totally ordered Abelian group and $u < e$, then the residuated-bounded lattice $\Gamma(\mathbf{G}, u) = \langle [u, e], \wedge, \vee, \circ, \backslash, /, e, u\rangle$ —where $x \circ y = xy \vee u$, $x\backslash y = x^{-1}y \wedge e$ and $x/y = xy^{-1} \wedge e$—is an MV-algebra. Conversely, if $\mathbf{L}$ is a totally-ordered MV-algebra, then there exists a totally ordered Abelian group with a strong order unit $u < e$ such that $\mathbf{L} \cong \Gamma(\mathbf{G}, u)$. This result was subsequently generalized for arbitrary Abelian ℓ-groups by D. Mundici [24] and recently for arbitrary ℓ-groups by A. Dvurečenskij [10]. It should be noted that all three authors have expressed their results in terms of the positive, rather than the negative, cone. Mundici and Dvurečenskij have also shown that the object assignment Γ can be extended to an equivalence between the category of MV-algebras (respectively, pseudo-MV-algebras), and the category with objects Abelian (respectively, arbitrary) ℓ-groups with a strong order unit, and morphisms ℓ-group homomorphisms that preserve the unit.

We generalize the concept of an MV-algebra in the setting of residuated lattices—by dropping integrality ($x \wedge e \approx x$), commutativity ($xy \approx yx$) and the existence of bounds—to a class that includes ℓ-groups, their negative cones, generalized Boolean algebras and