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Totally stably tame variables

Eric Edo

Département de Mathématiques Pures, Université Bordeaux I, 351, Cours de la Libération, 33405 Talence cedex, France

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Abstract

We improve the constructions of [Ann. Polon. Math. 76 (2001) 67] to obtain new automorphisms and variables of $A[x, y]$. We introduce the concept of totally stably tame variables. This allows us to prove that all our variables are stably tame. Comparing different notions of length, we show that some of our variables are not in Berson's class (cf. [J. Pure Appl. Algebra 170 (2002) 131] for definition).
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1. Introduction

Let n be an integer and let $x_1, \dots, x_n$ be n indeterminates (when $n = 2$, we set $x_1 = x$ and $x_2 = y$).

Throughout this paper, A denotes a domain. We denote by A^* the set of invertible elements of A, by $A^\times$ the set of nonzero elements of A, and by $\operatorname{qt} A = (A^\times)^{-1}A$ the quotient ring of A.

Let $A^{[n]} = A[x_1, \dots, x_n]$ be the ring of polynomials. If n and m are integers such that $n \leqslant m$, there exists a canonical inclusion $\theta_{n\to m} : A^{[n]} \to A^{[m]}$, but in this paper we think of $P \in A^{[n]}$ and $\theta_{n\to m}(P) \in A^{[m]}$ as two different objects. Let $\mathrm{GA}_n(A)$ be the automorphism group of the A-algebra $A^{[n]}$, if n and m are integers such that $n \leqslant m$, from

E-mail address: edo@math.u-bordeaux.fr.

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the inclusion $\theta_{n\to m} : A^{[n]} \to A^{[m]}$ we obtain an inclusion $\Theta_{n\to m} : \mathrm{GA}_n(A) \to \mathrm{GA}_m(A)$ (for $\sigma \in \mathrm{GA}_n(A)$, we define

$$\Theta_{n\to m}(\sigma)(x_i) = \begin{cases} \theta_{n\to m}\big(\sigma(x_i)\big) & \text{if } 1 \leqslant i \leqslant n, \\ x_i & \text{if } n+1 \leqslant i \leqslant m \end{cases}$$

). Furthermore, we identify $\mathrm{GA}_n(A)$ with its image by the canonical inclusion in $\mathrm{GA}_n(\mathrm{qt}\,A)$. The group $\mathrm{GA}_n(A)$ contains the two following subgroups: $\mathrm{Af}_n(A)$ the subgroup of affine (i.e., degree 1) automorphisms and $\mathrm{BA}_n(A)$ the subgroup of triangular automorphisms (i.e., $\sigma \in \mathrm{GA}_n(A)$ such that $\sigma(x_i) = a_i x_i + P_i$ with $a_i \in A^*$ and $P_i \in A[x_{i-1}, \dots, x_n]$). These two subgroups generate the subgroup $\mathrm{TA}_n(A)$ of *tame* automorphisms.

When $n = 2$ and if A is a field, every automorphism can be written as a product of affine automorphisms and triangular automorphisms in an almost unique way (cf. [6,7]):

Theorem 1 (Jung, van der Kulk, 1942–1953). *Let k be a field. We have*

$$\mathrm{GA}_2(k) = \mathrm{TA}_2(k) = \mathrm{Af}_2(k) * \mathrm{BA}_2(k),$$

where $*$ is the amalgamated product of $\mathrm{Af}_2(A)$ and $\mathrm{BA}_2(A)$ along their intersection.

One can express the equality $\mathrm{TA}_2(k) = \mathrm{Af}_2(k) * \mathrm{BA}_2(k)$ using the following definition.

Definition 1 (*Affine writing*). Let $\sigma \in \mathrm{TA}_n(A)$. We say that the sequence $(b_{l+1}, a_l, \dots, a_1, b_1)$ is an *affine writing* of σ if we have: $l \in \mathbb{N}$,

$$a_i \in \mathrm{Af}_n(A) \smallsetminus \mathrm{BA}_n(A) \quad \text{for all } i \in \{1, \dots, l\},$$
$$b_1, b_{l+1} \in \mathrm{BA}_n(A), \qquad b_i \in \mathrm{BA}_n(A) \smallsetminus \mathrm{Af}_n(A) \quad \text{for all } i \in \{2, \dots, l\}, \quad \text{and}$$
$$\sigma = b_{l+1} a_l b_l \dots a_1 b_1.$$

Corollary 1. *Let k be a field. Let $(b_{l+1}, a_l, \dots, a_1, b_1)$ and $(d_{m+1}, c_m, \dots, c_1, d_1)$ be two affine writings of $\sigma \in \mathrm{TA}_2(k)$. Then $l = m$ and there exists for all $i \in \{1, \dots, l\}$, $e_i, f_i \in \mathrm{BA}_2(k) \cap \mathrm{Af}_2(k)$ such that $c_i = f_i a_i e_i^{-1}$ and $d_i = e_i b_i f_{i-1}^{-1}$ for all $i \in \{1, \dots, l+1\}$ with $f_0 = e_{l+1} = \mathrm{Id}$.*

When $n = 2$ and if A is not a field, Corollary 1 implies that automorphisms which are not tame exist (cf. [8]).

Proposition 1 (Nagata, 1972). *We assume that A is not a field. Let $r \in A^{\times} \smallsetminus A^*$ and let*

$$\begin{cases} \sigma(x) = x + r^{-1}\big\{y^2 - \sigma(y)^2\big\}, \\ \sigma(y) = r^2 x + y + r y^2, \end{cases}$$

then $\sigma \in \mathrm{GA}_2(A) \smallsetminus \mathrm{TA}_2(A)$.