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Journal of Algebra 288 (2005) 137–211
JOURNAL OF Algebra
www.elsevier.com/locate/jalgebra

Cohen–Macaulay modules, (co)torsion pairs and virtually Gorenstein algebras

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Received 18 March 2004

Available online 7 April 2005

Communicated by Michel Van den Bergh

Dedicated to Claus Michael Ringel on the occasion of his sixtieth birthday

Abstract

We use torsion pairs in stable categories and cotorsion pairs in modules categories to study, in general infinitely generated, Cohen–Macaulay modules and (a generalization of) modules of finite projective or injective dimension over an Artin algebra. We concentrate our investigation to the study of *virtually Gorenstein* algebras which provide a common generalization of Gorenstein algebras and algebras of finite representation or Cohen–Macaulay type. This class of algebras on the one hand has rich homological structure and satisfies several representation/torsion theoretic finiteness conditions, and on the other hand it is closed under various operations, for instance derived equivalences and stable equivalences of Morita type. In addition virtual Gorensteinness provides a useful tool for the study of the Gorenstein Symmetry Conjecture and modified versions of the Telescope Conjecture for module or stable categories.

Keywords: Artin algebras; Cohen–Macaulay modules; Gorenstein rings; Stable categories; Covariantly, contravariantly finite and definable subcategories; Torsion pairs and cotorsion pairs; Triangulated categories; Compact objects; Telescope Conjecture; Gorenstein Symmetry Conjecture

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0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2005.02.022

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1. Introduction

Since the ubiquity fundamental paper of Bass [15] commutative Noetherian Gorenstein rings and Cohen–Macaulay modules are well established as central notions in commutative algebra bearing important connections with algebraic geometry. During the last decade there is an increasing growth of interest in non-commutative algebraic geometry, and, in this connection, several definitions of Gorensteinness were proposed by various authors in various settings. In particular in the representation theory of Artin algebras, Auslander–Reiten [9,11] introduced Gorenstein algebras as the Artin algebras with finite self-injective dimension from both sides, and they showed that much of the commutative theory carries over to Artin algebras. Also Happel [33] studied Gorenstein algebras in connection with Auslander–Reiten theory in derived categories. The class of Gorenstein algebras gains its importance from the fact that on the one hand it includes algebras with finite global dimension and self-injective algebras as special cases and on the other hand the finitely generated Cohen–Macaulay modules over them have rich homological structure and behave very well with respect to many natural operations and constructions at the level of the module or the derived category. In addition Gorenstein algebras have intimate connections with tilting theory and provide positive examples for many of the homological conjectures in the representation theory of Artin algebras.

Our aim in this paper is to study, in general infinitely generated, Cohen–Macaulay or CoCohen–Macaulay modules and modules of virtually finite projective or injective dimension over an arbitrary Artin algebra Λ. Using the terminology and notation of [22], see also [4,9], we denote by $\mathrm{CM}(\mathbf{P}_\Lambda)$, respectively $\mathrm{CoCM}(\mathbf{I}_\Lambda)$, the maximal subcategory of the category Mod-Λ of all right Λ-modules which admits the full subcategory $\mathbf{P}_\Lambda$, respectively $\mathbf{I}_\Lambda$, of projective, respectively injective, modules as an Ext-injective cogenerator, respectively Ext-projective generator. We call the modules in $\mathrm{CM}(\mathbf{P}_\Lambda)$, respectively $\mathrm{CoCM}(\mathbf{I}_\Lambda)$, *Cohen–Macaulay*, respectively *CoCohen–Macaulay*, *modules*. Then the full subcategory $\mathfrak{P}_\Lambda^{<\infty}$, respectively $\mathfrak{I}_\Lambda^{<\infty}$, of *modules of virtually finite projective*, respectively *injective, dimension* is defined to be the right, respectively left, Ext-orthogonal subcate-