Available online at www.sciencedirect.com

SCIENCE DIRECT

Journal of Algebra 284 (2005) 296–309

JOURNAL OF
Algebra

www.elsevier.com/locate/jalgebra

A note on tensor categories of Lie type E_9

Eric C. Rowell

Department of Mathematics, Indiana University, Bloomington, IN 47401 USA

Received 8 June 2004

Available online 15 September 2004

Communicated by Peter Littelmann

Abstract

We consider the problem of decomposing tensor powers of the fundamental level 1 highest weight representation V of the affine Kac–Moody algebra $\mathfrak{g}(E_9)$. We describe an elementary algorithm for determining the decomposition of the submodule of $V^{\otimes n}$ whose irreducible direct summands have highest weights which are maximal with respect to the null-root. This decomposition is based on Littelmann's path algorithm and conforms with the uniform combinatorial behavior recently discovered by H. Wenzl for the series E_N, $N \neq 9$.

1. Introduction

While a description of the tensor product decompositions for irreducible highest weight modules over affine algebras can be found in the literature (see, e.g., [1]), effective algorithms for computing explicit tensor product multiplicities are scarce. Some partial results in this direction have been obtained by computing characters (see, e.g., [2]) and by employing crystal bases (see, e.g., [5]) or the equivalent technique of Littelmann paths. In this note we look at the particular case of the affine Kac–Moody algebra associated to the Dynkin diagram E_9, with any eye towards extending the results of [6].

Let V be the irreducible highest weight representation of the $\mathfrak{g}(E_N)$, $N \geqslant 6$, with highest weight Λ_1 corresponding to the vertex in the Dynkin diagram furthest from the triple

E-mail address: errowell@indiana.edu.

0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2004.08.014

Table 1
Notation

α_i	ith simple root
Q	root lattice
Λ_i	ith fundamental weight
P	weight lattice
P_+	dominant weights
$\widehat{P}_+$	dominant weights (mod δ)
$\widehat{P}_+(n)$	level n dominant weights (mod δ)
$n(\lambda)$	level of λ
$P(\Lambda_1)$	weights of V
$W \cdot \Lambda_1$	maximal weights of V_{Λ_1}
Ω	set of straight weights
$P_+(V^{\otimes n})$	dominant weights of $V^{\otimes n}$
$[\lambda]_3$	least residue of $k(\lambda)$ (mod 3)
π_λ	path $t \to t\lambda$
W	(affine) Weyl group
$\mathcal{S}(k)$	level k initial weights
$\lambda \to \mu$	straight weight path

point. For $N \neq 9$, H. Wenzl [6] has found uniform combinatorial behavior for decomposing a certain submodule $V_{\text{new}}^{\otimes n}$ of $V^{\otimes n}$ using Littelmann paths [4]. These submodules have the property that each irreducible summand of $V_{\text{new}}^{\otimes n}$ appears in $V^{\otimes n}$ for the first time (for $N \leqslant 8$) or last time (for $N \geqslant 10$). The degeneracy of the invariant form was an obstacle to including the affine, $N = 9$ case.

We extend Wenzl's combinatorial description to the case $N = 9$ by finding submodules $\mathcal{M}_n$ analogous to his $V_{\text{new}}^{\otimes n}$. Specifically, we look at the (full multiplicity) direct sum of those submodules of $V^{\otimes n}$ whose highest weights have maximal null-root coefficient. Not surprisingly, these summands appear *only* in $V^{\otimes n}$. The particular utility of considering this submodule is that whereas decomposing the full tensor power $V^{\otimes n}$ into its simple constituents would require an infinite path basis, only a finite sub-basis (consisting of 200 straight paths) is needed to determine the decomposition of $\mathcal{M}_n$. Although this note was inspired by the results of [6], the module $\mathcal{M}_n$ appears so naturally that this case may shed some light on the combinatorial behavior described by Wenzl.

This paper is organized in the following way. In Section 2 we give the data and standard definitions for the Kac–Moody algebra $\mathfrak{g}(E_9)$. Section 3 is dedicated to summarizing the general technique of Littelmann paths, while in Section 4 we apply this technique to the present case and present some new definitions. Table 1 gives a glossary of notation for the reader's convenience. All the lemmas we prove are contained in Section 5, and the main theorem and algorithm they lead to is described and illustrated in Section 6. We briefly mention a possible application and a generalization in Section 7, as well as connections to Wenzl's results.

2. Notation and definitions

We begin by fixing a realization of the generalized Cartan matrix of $\mathfrak{g}(E_9)$ sometimes denoted in the literature by $\mathfrak{g}(E_8^{(1)})$. Observe that our realization is different than that of