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The K-theory of the flag variety and the Fomin–Kirillov quadratic algebra

Cristian Lenart [1]

Department of Mathematics and Statistics, State University of New York, Albany, NY 12222, USA

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Abstract

We propose a new approach to the multiplication of Schubert classes in the K-theory of the flag variety. This extends the work of Fomin and Kirillov in the cohomology case, and is based on the quadratic algebra defined by them. More precisely, we define K-theoretic versions of the Dunkl elements considered by Fomin and Kirillov, show that they commute, and use them to describe the structure constants of the K-theory of the flag variety with respect to its basis of Schubert classes.

1. Introduction

An important open problem in algebraic combinatorics is to describe combinatorially the structure constants for the cohomology of the flag variety Fl_n (that is, the variety of complete flags $(0 = V_0 \subset V_1 \subset \cdots \subset V_{n-1} \subset V_n = \mathbb{C}^n)$ in $\mathbb{C}^n$) with respect to its basis of Schubert classes. These structure constants are known as *Littlewood–Richardson coefficients*; a subset of them, consisting of the structure constants for the cohomology of a Grassmannian, are described by the classical Littlewood–Richardson rule. Fomin and Kirillov [2] proposed a new approach to the Littlewood–Richardson problem for $H^*(Fl_n)$

E-mail address: lenart@csc.albany.edu.
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based on a certain algebra with quadratic relations that they defined. In this paper, we extend Fomin and Kirillov's approach to the K-theory of Fl_n.

It is well known that the integral cohomology ring $H^*(Fl_n)$ and the Grothendieck ring $K^0(Fl_n)$ are both isomorphic to $\mathbb{Z}[x_1, \ldots, x_n]/I_n$, where I_n is the ideal generated by symmetric polynomials in $x_1, \ldots, x_n$ with constant term 0. In the cohomology case, the elements x_i are identified with the Chern classes of the dual line bundles L_i^*, where $L_i := V_i/V_{i-1}$ are tautological line bundles. In the K-theory case, we identify x_i with the K-theory Chern class $1 - 1/y_i$ of the line bundle L_i^*, where $y_i := 1/(1 - x_i)$ represents L_i in the Grothendieck ring.

One can define natural bases for $H^*(Fl_n)$ and $K^0(Fl_n)$ (over $\mathbb{Z}$) based on the CW-complex structure of Fl_n given by the (opposite) *Schubert varieties*. These are varieties X_w, which are indexed by permutations w in S_n, and which have complex codimension $l(w)$ (that is, the number of inversions in w). More precisely, if we think of Fl_n as SL_n/B, we let $X_w := \overline{B^- wB/B}$, where B and B^- are the subgroups of SL_n consisting of upper and lower triangular matrices. The *Schubert* and *Grothendieck polynomials* indexed by w, which are denoted by $\mathfrak{S}_w(x)$ and $\mathfrak{G}_w(x)$, are certain polynomial representatives for the cohomology and K-theory classes corresponding to X_w. These classes, which are denoted by σ_w and ω_w, form the mentioned natural bases for $H^*(Fl_n)$ and $K^0(Fl_n)$. Schubert and Grothendieck polynomials were defined by Lascoux and Schützenberger [4,5], and were studied extensively during the last two decades [3,6,8,10,11].

Both the Schubert polynomials $\mathfrak{S}_w(x)$ and the Grothendieck polynomials $\mathfrak{G}_w(x)$, for w in S_∞, form bases of $\mathbb{Z}[x_1, x_2, \ldots]$; here $S_\infty := \bigcup_n S_n$ under the usual inclusion $S_n \hookrightarrow S_{n+1}$. Hence we can write

$$\mathfrak{S}_u(x)\mathfrak{S}_v(x) = \sum_{w:\, l(w)=l(u)+l(v)} c_{uv}^w \mathfrak{S}_w(x), \qquad \mathfrak{G}_u(x)\mathfrak{G}_v(x) = \sum_{w:\, l(w)\geqslant l(u)+l(v)} c_{uv}^w \mathfrak{G}_w(x).$$

The notation is consistent since the structure constants corresponding to Schubert polynomials, which are known to be nonnegative for geometric reasons, are a subset of the structure constants corresponding to Grothendieck polynomials.

The simplest multiplication formula for Schubert polynomials is *Monk's formula*, which can be stated as follows:

$$x_p \mathfrak{S}_v(x) = -\sum_{\substack{1\leqslant i<p \\ l(vt_{ip})=l(v)+1}} \mathfrak{S}_{vt_{ip}}(x) + \sum_{\substack{i>p \\ l(vt_{pi})=l(v)+1}} \mathfrak{S}_{vt_{pi}}(x); \tag{1.1}$$

here t_{ij} is the transposition of i, j, and v is an arbitrary element of S_∞. In fact, Monk's formula expresses the product of $\mathfrak{S}_v(x)$ with a Schubert polynomial indexed by an adjacent transposition, which is equivalent to (1.1).

Similar formulas for Grothendieck polynomials were derived in [7]. Define the set $\Pi_p(v)$ to consist of all permutations

$$w = vt_{i_1 p} \ldots t_{i_r p} t_{p i_{r+1}} \ldots t_{p i_{r+s}},$$