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On extensions of modules

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Abstract

In this paper we study closely Yoneda's correspondence between short exact sequences and the Ext^1 group. We prove a main theorem which gives conditions on the splitting of a short exact sequence after taking the tensor product with R/I, for any ideal I of R. As an application, we prove a generalization of Miyata's theorem on the splitting of short exact sequences and we improve a proposition of Yoshino about efficient systems of parameters. We introduce the notion of sparse module and we show that $\mathrm{Ext}^1_R(M,N)$ is a sparse module provided that there are finitely many isomorphism classes of maximal Cohen–Macaulay modules having multiplicity the sum of the multiplicities of M and N. We prove that sparse modules are Artinian. We also give some information on the structure of certain Ext^1 modules.

Keywords: Extensions of modules; Rings of finite Cohen–Macaulay type

1. Introduction

Let R a Noetherian ring and M and N finitely generated R-modules. If $I \subset R$ is an ideal and $\alpha : 0 \to N \to X \to M \to 0$ is a short exact sequence, we denote by $\alpha \otimes R/I$ the sequence $0 \to N/IN \to X/IX \to M/IM \to 0$.

In the first part of this paper we prove the following theorem.

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Theorem 1.1. *Let R be a Noetherian ring. Suppose $\alpha \in I \operatorname{Ext}^1_R(M, N)$ is a short exact sequence of finitely generated modules. Then $\alpha \otimes R/I$ is a split exact sequence.*

An immediate corollary is an extension of Miyata's theorem [1], which gives a necessary and sufficient condition on the splitting of short exact sequences:

Theorem 1.2. *Let R be a Noetherian ring. Let $\alpha : 0 \to N \to X_\alpha \to M \to 0$ and $\beta : 0 \to N \to X_\beta \to M \to 0$ be two short exact sequences. If X_α is isomorphic to X_β and $\beta \in I \operatorname{Ext}^1_R(M, N)$ for some ideal $I \subset R$, then $\alpha \otimes R/I$ is split exact.*

In the second part of the paper, we give some applications of Theorem 1.1 to efficient systems of parameters and, more in particular, to the structure of $\operatorname{Ext}^1_R(M, N)$ for rings of finite Cohen–Macaulay type.

Recall that a finitely generated R-module is said to be maximal Cohen–Macaulay if the $\operatorname{depth}(M) = \dim(R)$. We also say that R has finite Cohen–Macaulay type if there are a finite number of isomorphism classes of indecomposable maximal Cohen–Macaulay modules. Auslander [2] proved that if R is a complete local ring of finite Cohen–Macaulay type then the length of $\operatorname{Ext}^1_R(M, N)$ is finite, where M and N are maximal Cohen–Macaulay modules. Recently, Huneke and Leuschke [3] generalized this theorem to the non-complete case. A different proof is given as an application of Theorem 1.1.

In Section 4 we introduce the notion of sparse modules. More specifically, suppose that depth $R \geqslant 1$; then we say that a module is sparse if there are a finite number of submodules of the form xM where x is a non-zero-divisor on R. We prove several properties for sparse modules. In particular, we show that sparse modules are Artinian and that $\operatorname{Ext}^1_R(M, N)$ is sparse if M and N are maximal Cohen–Macaulay modules over a ring of finite Cohen–Macaulay type.

In the last section, we are able to give more information about the structure of $\operatorname{Ext}^1_R(M, N)$. In particular, we give an explicit bound for the power of the maximal ideal which kills the Ext module, depending on the number of isomorphism classes of maximal Cohen–Macaulay modules of multiplicity the sum of the multiplicities of M and N. Our bound improves the one given in [3]. Finally, we use the developed techniques to show that $\operatorname{Ext}^1_R(M, N)$ is a cyclic module, under certain conditions.

2. Main theorem and Miyata's theorem

We recall the theorem due to Miyata [1] on the splitting of short exact sequences.

Theorem 2.1 (Miyata). *Let $(R, \mathbf{m})$ be a local Noetherian ring and let*

$$\alpha: \quad 0 \longrightarrow N \xrightarrow{\ i\ } X_\alpha \xrightarrow{\ \pi\ } M \longrightarrow 0$$

be a short exact sequence of finitely generated R-modules. Then, α is a split exact sequence if and only if X_α and $M \oplus N$ are isomorphic.