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On some subalgebras of von Neumann algebras with analyticity

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Abstract

Let (M, α, G) be a covariant system on a locally compact Abelian group G with the totally ordered dual group $\hat{G}$ which admits the positive semigroup $\hat{G}_+$. Let $H^\infty(\alpha)$ be the associated analytic subalgebra of M; i.e. $H^\infty(\alpha) = \left\{x \in M \mid \mathrm{Sp}_\alpha(x) \subseteq \hat{G}_+\right\}$. Let $N \rtimes_\theta \hat{G}_+$ be the analytic crossed product determined by a covariant system $(N, \theta, \hat{G})$. We give the necessary and sufficient condition that an analytic subalgebra $H^\infty(\alpha)$ is isomorphic to an analytic crossed product $N \rtimes_\theta \hat{G}_+$ related to Landstad's theorem. We also investigate the structure of σ-weakly closed subalgebra of a continuous crossed product $N \rtimes_\theta \mathbb{R}$ which contains $N \rtimes_\theta \mathbb{R}_+$. We show that there exists a proper σ-weakly closed subalgebra of $N \rtimes_\theta \mathbb{R}$ which contains $N \rtimes_\theta \mathbb{R}_+$ and is not an analytic crossed product. Moreover we give an example that an analytic subalgebra is not a continuous analytic crossed product using the continuous decomposition of a factor of type $\mathrm{III}_\lambda (0 \leqslant \lambda < 1)$.

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1. Introduction

Let G be a locally compact Abelian group with the totally ordered dual group $\hat{G}$ which has a positive semigroup $\hat{G}_+$. Let M(resp. N) be a von Neumann algebra and let α(resp. β) be a σ-weakly continuous action of G on M(resp. $\hat{G}$ on N). The analytic subalgebra $H^\infty(\alpha)$ determined by α is the σ-weakly closed subalgebra of M defined by

$$H^\infty(\alpha) = \left\{ x \in M \mid \mathrm{Sp}_\alpha(x) \subseteq \hat{G}_+ \right\},$$

where $\mathrm{Sp}_\alpha(\cdot)$ is the Arveson spectrum(cf. [5]). The analytic crossed product $N \rtimes_\beta \hat{G}_+$ determined by N and β is the σ-weakly closed subalgebra of the crossed product $N \rtimes_\beta \hat{G}$. (These precise definitions give to the next section.) Roughly speaking, the analytic crossed products stand in the same relation to the crossed products as the Hardy algebras $H^\infty(G)$, the space of all functions of analytic type which belongs to $L^\infty(G)$, stand in relation to $L^\infty(G)$. These algebras provide a very interesting generalization to the noncommutative setting of certain well-known classes of function algebras and so there are many researches for these algebras; maximality, invariant subspace structure, factorization problem and so on. In this paper we study the structure of these algebras and subalgebras which contains an analytic crossed product.

In the next section we establish notation and discuss the isomorphism of these algebras. In the case when $\hat{G} = \mathbb{R}$ or $\mathbb{Z}$ it is well known that any analytic crossed product $N \rtimes_\beta \hat{G}_+$ is the analytic subalgebra $H^\infty(\tilde{\beta})$ of $N \rtimes_\beta \hat{G}$ determined by the dual action $\tilde{\beta}$ of β. On the other hand, in the case $G = \mathbb{T}$, $\hat{G} = \mathbb{Z}$ and $\hat{G}_+ = \mathbb{Z}_+$, any analytic subalgebras (also any analytic crossed products) are maximal subdiagonal algebras with respect to a canonical conditional expectation in the sense of Arveson [1]. In [6] it was studied which subdiagonal algebras are analytic crossed products. And, using the theory of invariant subspaces, they proved that if a version of the Beurling, Lax, Halmos theorem is valid in a subdiagonal algebra, then it must be a discrete analytic crossed product. Motivated by these fact, we consider the following problem:

Problem 1. Which analytic subalgebras are analytic crossed products?

There is an important theorem characterises those von Neumann algebras which are crossed products by a continuous action of G (see [17, Theorem 19.9]). This result was obtained independently by Landstad [3] and Connes and Takesaki [2]. Applying this result, we shall give the partial answer of Problem 1.(Theorem 2.2) Moreover we shall show that every analytic crossed product is an analytic subalgebra in the general case (Proposition 2.3).

In Section 3, we study the structure of subalgebras of crossed product $N \rtimes_\beta \mathbb{R}$ that contains $N \rtimes_\beta \mathbb{R}_+$ related to the maximality of $N \rtimes_\beta \mathbb{R}_+$. We know the following results which relates to analytic subalgebras. When $H^\infty(\alpha)$ is not maximal it was shown, in some cases, that the σ-weakly closed subalgebras of M that contains $H^\infty(\alpha)$ have special properties. For example, it was shown in [15] that, when α is periodic, every