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Journal of Functional Analysis 222 (2005) 306–384

JOURNAL OF Functional Analysis

www.elsevier.com/locate/jfa

A class of Banach spaces with few non-strictly singular operators

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Received 7 April 2004; received in revised form 27 October 2004; accepted 4 November 2004
Communicated by G. Pisier
Available online 22 December 2004

Abstract

We construct a family $(\mathfrak{X}_\gamma)$ of reflexive Banach spaces with long (countable as well as uncountable) transfinite bases but with no unconditional basic sequences. The method we introduce to achieve this allows us to considerably control the structure of subspaces of the resulting spaces as well as to precisely describe the corresponding spaces on non-strictly singular operators. For example, for every pair of countable ordinals γ, β, we are able to decompose every bounded linear operator from $\mathfrak{X}_\gamma$ to $\mathfrak{X}_\beta$ as the sum of a diagonal operator and an strictly singular operator. We also show that every finite-dimensional subspace of any member $\mathfrak{X}_\gamma$ of our class can be moved by and $(4+\varepsilon)$-isomorphism to essentially any region of any other member $\mathfrak{X}_\delta$ or our class. Finally, we find subspaces X of $\mathfrak{X}_\gamma$ such that the operator space $\mathcal{L}(X, \mathfrak{X}_\gamma)$ is quite rich but any bounded operator T from X into X is a strictly singular pertubation of a scalar multiple of the identity.

MSC: primary 46B20; 03E05; Secondary 46B15; 46B28; 03E02

Keywords: Unconditional basic sequences; Transfinite Schauder bases; Strictly singular operators; Diagonal operators; Hereditarily indecomposable Banach spaces; Reflexive Banach spaces; Uncountable codings

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[1]Supported by a European Community Marie Curie Fellowship.

0022-1236/$ - see front matter
doi:10.1016/j.jfa.2004.11.001

0. Introduction

The original motivation for this paper is based on the natural question left open by the Gowers–Maurey solution of the unconditional basic sequence problem for Banach spaces [12]. Recall that Gowers and Maurey have constructed a Banach space X with a Schauder basis $(e_n)_n$ but with no unconditional basic sequence. Thus, while every infinite dimensional Banach space contains a sequence $(x_n)_n$ which forms a Schauder basis for its closure $Y = \overline{\langle x_n \rangle_n}$, meaning that every vector of Y has a unique representation $\sum_n a_n x_n$, one may not be able to get such $(x_n)_n$ such that the sums $\sum_n a_n x_n$ converge unconditionally whenever they converge. The fundamental role of Schauder basis and the fact that the notion is very much dependent on the order lead to the natural variation of the notion, the definition of transfinite Schauder basis $(x_\alpha)_{\alpha<\gamma}$, where vectors of X have a unique representations as sums $\sum_{\alpha<\gamma} a_\alpha x_\alpha$. In fact, as it will be clear from some results in this paper, considering a transfinite Schauder basis, even if one knows that X has an ordinary Schauder basis, can be an advantage. Thus, the natural question which originated the research of this paper asks whether one can have Banach spaces with long (even of uncountable length) Schauder bases but with no unconditional basic sequence. There is actually a more fundamental reason for asking this question. As noticed originally by W. B. Johnson, the Gowers–Maurey space X is hereditarily indecomposable which in particular yields that the space of operators on X is very small in the sense that every bounded linear operator on X can be written as $\lambda \mathrm{Id}_X + S$, where S is a strictly singular operator. On the other hand, if X has a transfinite Schauder basis $(e_\alpha)_{\alpha<\gamma}$ of length, say, $\gamma = \omega^2$, it could no longer have so small an operator space as projections on infinite intervals $\overline{\langle e_\alpha \rangle_{\alpha \in I}}$ are all (uniformly) bounded. Thus one would like to find out the amount of control on the space of non-strictly singular operators that is possible in this case. In fact, our solution of the transfinite variation of the unconditional basic sequence problem has led us to many other new questions of this sort, has forced us to introduce several new methods to this area, and has revealed several new phenomena that could have been perhaps difficult to discover by working only in the context of ordinary Schauder bases.

To see the necessity for a new method we repeat that our first goal here is to construct a Banach space $\mathfrak{X}_{\omega_1}$ with a transfinite Schauder basis $(e_\alpha)_{\alpha<\omega_1}$ with no unconditional basic sequence as well as to understand its separable initial segments $\mathfrak{X}_\gamma = \overline{\langle e_\alpha \rangle_{\alpha<\gamma}}$. The original Gowers–Maurey method for preventing unconditional basic sequences is to force the unconditional constants of initial finite-dimensional subspaces, according to the fixed Schauder basis, grow to infinity. Since initial finite-dimensional subspaces according to our transfinite Schauder basis $(e_\alpha)_{\alpha<\omega_1}$ are far from exhausting the whole space, their method will not work here. It turns out that in order to impose the conditional structure on our space(s) $\mathfrak{X}_\gamma$ ($\gamma \leqslant \omega_1$) we needed to import a tool from another area of mathematics, a rather canonical semi-distance function ϱ on the space ω_1 of all countable ordinals [26]. What ϱ does in our context here is to essentially identify the structure of finite-dimensional subspaces of various $\mathfrak{X}_\gamma$'s which globally are of course very much different, since for example $\mathfrak{X}_\omega$ is hereditarily indecomposable while, say, $\mathfrak{X}_{\omega^2}$ has a rich space of non-strictly singular operators.