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A Fourier view on the R-transform and related asymptotics of spherical integrals

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Abstract

We estimate the asymptotics of spherical integrals of real symmetric or Hermitian matrices when the rank of one matrix is much smaller than its dimension. We show that it is given in terms of the R-transform of the spectral measure of the full rank matrix and give a new proof of the fact that the R-transform is additive under free convolution. These asymptotics also extend to the case where one matrix has rank one but complex eigenvalue, a result related with the analyticity of the corresponding spherical integrals.

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1. Introduction

1.1. General framework and statement of the results

In this article, we consider the spherical integrals

$$I_N^{(\beta)}(D_N, E_N) := \int \exp\{N\mathrm{tr}(UD_NU^*E_N)\}\, dm_N^{\beta}(U),$$

where $m_N^{(\beta)}$ denote the Haar measure on the orthogonal group $\mathcal{O}_N$ when $\beta = 1$ and on the unitary group $\mathcal{U}_N$ when $\beta = 2$, and D_N, E_N are $N \times N$ matrices that we can assume diagonal without loss of generality. Such integrals are often called, in the physics literature, Itzykson–Zuber or Harish-Chandra integrals. We do not consider the case $\beta = 4$ mostly to lighten the notations.

The interest for these objects goes back in particular to the work of Harish-Chandra ([12,13]) who intended to define a notion of Fourier transform on Lie algebras. They have been then extensively studied in the framework of so-called matrix models that are related to the problem of enumerating maps (after [14], it has been developed in physics for example in [27,17] or [19], in mathematics in [5] or [9]; a very nice introduction to these links is provided in [28]). The asymptotics of the spherical integrals needed to solve matrix models were investigated in [11]. More precisely, when D_N, E_N have N distinct real eigenvalues $(\theta_i(D_N), \lambda_i(E_N))_{1\leqslant i\leqslant N}$ and the spectral measures $\hat{\mu}_{D_N}^N = \frac{1}{N}\sum \delta_{\theta_i(D_N)}$ and $\hat{\mu}_{E_N}^N = \frac{1}{N}\sum \delta_{\lambda_i(E_N)}$ converge, respectively, to μ_D and μ_E, it is proved in Theorem 1.1 of [11] that

$$\lim_{N\to\infty} \frac{1}{N^2} \log I_N^{(\beta)}(D_N, E_N) = I^{(\beta)}(\mu_D, \mu_E) \tag{1}$$

exists under some technical assumptions and a (complicated) formula for this limit is given.

In this paper, we investigate different asymptotics of the spherical integrals, namely the case where one of the matrix, say D_N, has rank much smaller than N.

Such asymptotics were also already used in physics (see [18], where they consider replicated spin glasses, the number of replica there being the rank of D_N) or stated for instance in [5, Section 1], as a formal limit (the spherical integral being seen as a series in θ when $D_N = \mathrm{diag}(\theta, 0, \ldots, 0)$ whose coefficients are converging as N goes to infinity). However, to our knowledge, there is no rigorous derivation of this limit available in the literature. We study this problem by use of large deviations techniques here. The proofs are however rather different from those of [11]; they rely on large deviations for Gaussian variables and not on their Brownian motion interpretation and stochastic analysis as in [11].

Before stating our results, we now introduce some notations and make a few remarks.

Let $D_N = \mathrm{diag}(\theta, 0, \ldots, 0)$ have rank one so that

$$I_N^{(\beta)}(D_N, E_N) = I_N^{(\beta)}(\theta, E_N) = \int e^{\theta N (UE_NU^*)_{11}}\, dm_N^{(\beta)}(U). \tag{2}$$